STATISTICAL INERTIAL NAVIGATION

by

Gerald E. Steinker

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ABSTRACT

This thesis is concerned with inertial navigation of vehicles during short time intervals for which the predominant accelerations are non-gravitational. Due to the high time correlations of random errors associated with an inertial measurement unit, it is possible to assume accelerometer measurements which are perfect except for random constant error coefficients. Statistical estimation theory is applied to navigation systems with assumed perfect measurements and the estimation errors associated with such a statistical navigation system are compared with errors derived from conventional deterministic navigation systems.

The design and effectiveness of the statistical navigation system depend on the number of independent white noise elements driving the non-gravitational or specific force accelerations. If this number is equal to or less than the number of measurements observed, the estimation errors of the statistical system are shown to asymptotically approach zero.

Development of a statistical navigation system for an Apollo re-entry mission is presented. The nature of the sensitivity of acceleration variations to some components of white noise suggests a statistical navigation system containing two independent filters which are employed alternately as the vehicle roll angle is altered. Such a system is shown to be effective in immediately reducing initial estimation errors. A simplified single filtering navigation system is obtained with the inclusion of arbitrary additive white noise in the measurements. This system reveals a more continuous but equally dramatic reduction of initial estimation errors. Both systems show a marked improvement in navigation accuracy over the conventional deterministic approach.

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CHAPTER I

INTRODUCTION

Steering or controlling a vehicle to some prescribed destination can be performed by the operation, in cascade, of four subsystems:

1. A measurement system which gathers information describing the state of the vehicle.
2. A navigation system which processes the measurements to determine the present location and course of the vehicle.
3. A guidance system which compares the present course with one which will intersect the destination and generates maneuver commands to steer the vehicle on the intersecting course.
4. A control system which controls the vehicle to comply with the commands received from the guidance system.

This thesis is concerned with the gathering and processing of information to determine the present state of the vehicle, i.e., the measurement and navigation systems. The form of the navigation system depends on the type of information received from the measurement system, which, in turn, depends on the surrounding environment in which the vehicle is traveling. For example, travel on land allows a measurement system to observe recognizable land sites or road markings which are used by the navigator through comparison with reference markings on a map to determine position of the vehicle.

Without the availability of landmarks, the measurement system must depend upon other sources of information within the environment. These sources could include radiation links through radar to known positions, or sightings of celestial bodies and constellations. Celestial sightings were used extensively in early sea navigation and are employed today for navigation in "free fall" space flight.

The development of inertial measurement devices provided a new source of information to navigation systems. The natural property of the inertial measurement unit (IMU) to maintain a known reference orientation and to sense specific force accelerations allows for a self contained navigation system aboard vehicles such as ships, aircraft and spacecraft operating in an environment of known gravitational field. Vehicle position and velocity can be obtained through integration of the total acceleration received from the IMU and from knowledge of the gravitational acceleration acting on the vehicle.

Associated with any measurement system is an inherent error or uncertainty which forbids the exact determination of position by the navigator. Additional errors might be introduced by the navigator itself while converting the measurement data into vehicle position and velocity components. In the case of inertial navigation systems, initial condition uncertainties play a large role in navigation errors throughout the mission.

If the navigator employs only the measurement data in determining the state of the vehicle, it must be content to accept these errors. However, if the navigator has available any additional information concerning the environment in which the vehicle is traveling, the dynamics of the vehicle itself, and/or the characteristics of the measurement system, which would allow it to predict what the measurement should be, it could compare this prediction with the actual measurement. If the predicted and measured values do not agree, then the navigator could choose some value in between the two depending upon which value it considered more likely to be correct.

This method of obtaining more accurate estimates of vehicle position has been used quite extensively in the past. Inertial navigation systems aboard vehicles moving at low speeds near the surface of the earth recognize the fact that the primary specific force acting on the vehicle is the gravitational force directed normal to the earth's surface, and that the earth rotates at a known constant rate. Hence, by forcing the IMU platform to maintain a level position normal to the measured specific force and to rotate at the earth rate, improved estimates could be made of the vehicles latitude and azimuth through comparison of the orientation of this platform with a fixed inertial coordinate frame.

A more systematic method of improving the knowledge of the state of the vehicle is found through the use of optimal estimation theory in which the entire system is treated in a statistical sense. Statistical estimation theory was first developed by Wiener(1) and later set into a more general "state space" context by Kalman.(2) This theory uses knowledge of the vehicle dynamics, as well as the statistical properties of random forcing functions inherent in the vehicle dynamics and measurement system, to design a filter whose output is an estimate of the vehicle position and velocity. The mathematical framework of the filter was presented in an historic paper by Kalman and Bucy.[3]

Many persons have studied navigation systems incorporating statistical estimation theory. Such navigation systems first became operational through the Minivar Program[4][5] for orbit prediction of earth satellites. Navigation aboard a circumlunar vehicle was studied by Smith, et. al.[6] and later by Farrell[7] for measurements from radar and celestial data.

An independent derivation of statistical estimation theory was provided by Battin[8] for achieving "maximum likelihood" navigation from discrete celestial sightings during midcourse space flight.

Until recently, statistical estimation was employed only to "free fall" space trajectories for which the vehicle dynamics could be modeled quite accurately from knowledge of the surrounding gravitational fields. Its application was carried to navigation of vehicles near or on the earth by Fagin[9] and Brown and Friest[10] using position fixes and velocity log data and later by Brock[11] with the use of inertial measurement data. These studies assumed the primary specific force on the vehicle to be the negative of the gravitational force and the primary motion of the vehicle with respect to an inertial frame to be the earth's rotation. Hence these studies were applicable to vehicles maintaining essentially constant low speed near the earth.

Significantly new and different navigation problems arise when the vehicle is accelerated primarily by forces other than gravitational or apparent forces. This phase of high acceleration is usually a small portion of an overall mission such as ascent, descent, orbital change, or re-entry of a space craft or gross maneuvers of a sea-going vessel. Hence it is usually of short time duration and the primary forces are propulsive, aerodynamic or hydrodynamic.

Navigation during these maneuvers has been accomplished in the past through basically deterministic methods with IMU measurements or radar tracking data. The high accuracy of inertial measurement devices provides excellent knowledge of the specific force accelerations. The accuracy of the navigation, however, is greatly impaired by the uncertainty in initial knowledge of the state of the vehicle, which cannot be corrected by deterministic integration of the accelerations. Radar data provides considerable aid in reducing these uncertainties but is limited by its availability during some mission phases.

Few studies have been directed towards the extension of statistical estimation theory to navigation during these high acceleration maneuvers. Wagner(12) has investigated the employment of the Kalman-Bucy filter for accurate prediction of re-entry orbits with the use of measurements from both on-board inertial accelerometers and ground-based tracking stations. In this study the random measurement errors for both systems were assumed to be additive white noise. It has been found that the white noise assumption is quite valid in describing radar measurement uncertainties. However, intensive study of inertial measurement systems reveal that the random error coefficients describing errors inherent in these systems are highly correlated in time and hence, during relatively short intervals of time, could better be represented as random constants.

The development of statistical estimation theory for measurements containing colored noise by Bryson and Johansen[13] and by Deyst[14] has paved the way for study of navigation systems in which the information received is essentially perfect. The application of this perfect measurement estimation theory could be applied in conjunction with the inertial measurement system to provide for a more realistic and accurate navigation system during periods of finite specific force accelerations. This is the basis for the present research described within this thesis.

1.1 Thesis Objective

This thesis develops a statistical navigation system to be employed with an inertial measurement unit for vehicles encountering accelerations predominately due to non-gravitational specific forces, and shows the applicability of such navigation systems to a representative mission phase of atmospheric re-entry.

Chapter II contains the theoretical development of the navigation system which is statistical in nature and a comparison with the conventional (deterministic) system. The application of statistical inertial navigation to atmospheric re-entry is illustrated in Chapter III. Chapter IV presents numerical results obtained from a computer simulation of a typical Apollo re-entry mission and shows a marked improvement in accuracy of the statistical navigator over the conventional navigation scheme. Conclusions and recommendations derived from this research are discussed in Chapter V.

CHAPTER II

DEVELOPMENT OF INERTIAL NAVIGATION SYSTEMS

This chapter develops an inertial navigation system for vehicles acted upon by specific forces over short periods of time. The estimation errors associated with a deterministic and statistical navigation system are compared.

Within this chapter, navigation will be considered only with respect to an inertial reference frame. This allows the presentation of meaningful results without the complexities introduced by transformation to an accelerating or rotating frame of reference. The application of re-entry navigation considered in Chapter III will study navigation in a non-inertial coordinate system.

Navigation will also be confined to the determination of position and velocity of a vehicle with the vehicle assumed as a mass particle. The attitude of the vehicle will not be considered in this thesis.

The development of an inertial navigation system presupposes knowledge of the operation of an inertial measurement unit. Hence, we begin with a brief description and error analysis of this unit.

2.1 Description of IMU

An inertial measurement unit (IMU) is composed of accelerometers mounted orthogonally on a platform which is controlled by gyros to maintain a fixed orientation in inertial space.

An accelerometer may be viewed, for our present purpose, as a linear mass-spring combination. When subjected to an external force, the mass will be deflected by an amount proportional to the magnitude of the acceleration caused by this force. However, an accelerometer will not give an indication of the total acceleration of the vehicle, but rather the difference between the true acceleration and the field or apparent force accelerations. This difference is called the acceleration due to the <u>specific forces</u> acting on the vehicle, e.g., propulsive, aerodynamic, and bouyant forces.

The errors in an IMU may be attributed to two independent sources: misalignment of the stable platform from the expected true inertial orientation and errors in accelerometer readings. Misalignment is caused by drifting of the gyros from their preset orientation and by an initial misalignment of the stable platform. Gyro drift rate, according to Laning[15], is well approximated by a quadratic dependence on the specific force acceleration and may be written as

$$\dot{\Delta}_i = w_{do_i} + \underline{w}_{dl_i}^T \underline{a}_{sf} + \underline{a}_{sf}^T W_{d_i} \underline{a}_{sf} \qquad (2.1)$$

where Δ_i is the drift angle of the i^{th} gyro element,

$\underline{a}_{sf}$ is the vector specific force acceleration,

w_{do}, $\underline{w}_{dl}$, and W_d are, respectively, the bias, acceleration-sensitive, and acceleration squared-sensitive error coefficients for the i^{th} gyro element,

and where $\Delta_i(t_o)$ is the initial misalignment angle about the i^{th} gyro input axis.

If we consider the drift component of each gyro as a component of a three dimensional drift vector, $\underline{\Delta}$, we may determine the error in measured acceleration due to the stable platform misalignment as the vector product of $\underline{\Delta}$ with the measured acceleration vector

$$\delta \underline{a}_{gyro} = \underline{a}_{sf} \times \underline{\Delta}$$

or, simply as

$$\delta \underline{a}_{gyro} = D_1 \underline{\Delta} \tag{2.2}$$

The error in accelerometer reading is also well approximated by a quadratic dependence on the specific force acceleration as

$$\delta a_{accel._i} = w_{ao_i} + \underline{w}_{al_i}^T \underline{a}_{sf} + \underline{a}_{sf}^T W_{a_i} \underline{a}_{sf} \tag{2.3}$$

where $\delta a_{accel._i}$ is the error in the i^{th} accelerometer and where w_{ao}, $\underline{w}_{al}$, and W_a are the error coefficients relating the dependence of this error on the measured accelerations.

The total error in acceleration information received from the IMU may then be described as

$$\delta \underline{a}_m = D_1 \underline{\Delta} + \delta \underline{a}_{accel.} \tag{2.4}$$

All of the error coefficients, $w_{do} \ldots W_a$ are, in general, random variables with non-zero mean values. However, with adequate testing, calibration, and compensation, it is possible to make the mean values zero.

The statistical properties of these error coefficients have been the concern of much research during the past several years.(16)(17)(18)(19) These studies have shown that the error coefficients w_{do}, $\underline{w}_{dl}$, and W_d may be considered as random constants over time periods of less than one to three hours.

If we consider all the error coefficients ($w_{do} \cdots W_{a_i}$) contained as individual elements in a random constant vector, $\underline{\omega}$ (the inclusion of all 13 coefficients each for three gyros and three accelerometers would imply 78 elements in the $\underline{\omega}$ vector), equations (2. 3) and (2. 1) for accelerometer error and drift rate may then be written as linear combinations of the vector $\underline{\omega}$ as

$$
\begin{aligned}
\delta \underline{a}_{accel.} &= D_2 \underline{\omega} \\
\dot{\underline{\Delta}} &= D_3 \underline{\omega}
\end{aligned}
\tag{2. 5}
$$

where the matrices D_2 and D_3 would each be of dimension 3x78 and would take the form

$$
D_2 = \begin{pmatrix} [\quad] & O & O & O & O & O \\ O & [\quad] & O & O & O & O \\ O & O & [\quad] & O & O & O \end{pmatrix}
$$

$$
D_3 = \begin{pmatrix} O & O & O & [\quad] & O & O \\ O & O & O & O & [\quad] & O \\ O & O & O & O & O & [\quad] \end{pmatrix}
$$

with each O being a 13 element row vector of zeros and [ ] being a 13 element row vector of functions of $\underline{a}_{sf}$.

With the aid of a new vector $\underline{b}$ defined as

$$\underline{b} = \begin{Bmatrix} \underline{\Delta} \\ \hdashline \underline{\omega} \end{Bmatrix} \tag{2.6}$$

we may then represent the IMU acceleration error, (2.4), as

$$\delta \underline{a}_m = E\, \underline{b} \tag{2.7}$$

where the matrix, E, is defined as

$$E = \left[D_1 \,\middle\vdots\, D_2 \right] \tag{2.8}$$

and where

$$\dot{\underline{b}} = D_o \underline{b} = \left[\begin{array}{c:c} O & D_3 \\ \hdashline O & O \end{array} \right] \underline{b} \tag{2.9}$$

The matrices D_o through D_3 depend upon the specific force acceleration time histories of the vehicle from initial alignment of the platform. The vector $\underline{b}$ will thus be a random vector with assumed known statistical characteristics represented by the mean

$$\mathcal{E} [\, \underline{b}(t) \,] = \underline{0}$$

and by the covariance matrix

$$P_b(t) = \mathcal{E} [\, \underline{b}(t)\, \underline{b}(t)^T \,] \tag{2.10}$$

2.2 Deterministic Navigation with IMU Measurements

The equations of motion of a vehicle considered as a mass particle may be obtained from Newton's laws as

$$\begin{aligned} \dot{\underline{v}} &= \underline{a} \\ \dot{\underline{r}} &= \underline{v} \end{aligned} \tag{2.11}$$

The total acceleration, $\underline{a}$, of a vehicle, is composed of specific force accelerations, $\underline{a}_{sf}$, measured by the IMU, and of gravitational accelerations, $\underline{g}$. The specific force accelerations will, in general, be a function of the position and velocity of the vehicle, and time. Since no information is received from the measurements concerning the gravitational acceleration, this term must be derived from prior knowledge of the gravitational field and from the present estimate of the vehicle's position in this field. If such knowledge is available, determination of vehicle position and velocity may be obtained as the solution of the differential equations

$$\begin{aligned} \dot{\overline{\underline{v}}}(t) &= \underline{a}_m(\underline{v}, \underline{r}, t) + \underline{g}(\overline{\underline{r}}) \\ \dot{\overline{\underline{r}}}(t) &= \overline{\underline{v}}(t) \end{aligned} \tag{2.12}$$

where $\overline{\underline{r}}(t)$ and $\overline{\underline{v}}(t)$ represent the estimate of the vehicle position and velocity at time, t, and $\underline{a}_m$ is the acceleration information received from the IMU.

In order to study the errors in this deterministic navigation scheme, we consider linear perturbations about a nominal path of the vehicle with the use of a first order Taylor series expansion (Hansen, et. al., [20] show that such linearization gives acceptable results even for a highly non-linear re-entry trajectory). The actual path of the vehicle may thus be represented as

$$\underline{v}(t) = \underline{v}_o(t) + \delta\underline{v}(t)$$
$$\underline{r}(t) = \underline{r}_o(t) + \delta\underline{r}(t)$$

where the subscript o refers to the nominal path and $\delta\underline{r}$ and $\delta\underline{v}$ describe small perturbations about the nominal path which we consider as random variables. In the same manner, we may describe our estimate of the actual path as

$$\bar{\underline{v}}(t) = \underline{v}_o(t) + \delta\bar{\underline{v}}(t)$$
$$\bar{\underline{r}}(t) = \underline{r}_o(t) + \delta\bar{\underline{r}}(t)$$

With the linearity assumption, we may express the differential equations of the variations in actual position and velocity as

$$\delta\dot{\underline{v}} = \delta\underline{a} = \delta\underline{a}_{sf}(\underline{v}, \underline{r}, t) + \delta\underline{g}(\underline{r})$$
$$\delta\dot{\underline{r}} = \delta\underline{v}$$

We will assume that the model of the gravitational field is sufficiently accurate to describe the effects of earth oblateness and of the gravitational fields of other nearby celestial bodies. (The result of intentionally ignoring these effects is deterministic in nature and could be examined independent of the present statistical analysis.) Variations from the known nominal gravitational accelerations will then be caused by random gravitational anomalies as well as to the random perturbations in the radial position, $\delta\underline{r}$, within the known gravitational field.

An investigation of the random gravitational anomalies might allow us to create a representative colored noise model of these variations, which, in turn, could be obtained from white noise through the use of shaping filter techniques. For purposes of simplicity, however, we will cautiously consider these random variations to be of such high frequency that they could be represented as white noise. With these assumptions, then, we can express $\delta \underline{g}$ as

$$\delta \underline{g}(t) = \frac{\partial \underline{g}}{\partial \underline{r}} \delta \underline{r} + \underline{u}_g = \Gamma \delta \underline{r} + \underline{u}_g \tag{2.13}$$

where $\underline{u}_g$ is white noise with

$$\mathcal{E}[\underline{u}_g] = \underline{0}$$

and

$$\mathcal{E}[\underline{u}_g(t)\, \underline{u}_g(\tau)^T] = Q_g(t)\, \delta(t-\tau) \tag{2.14}$$

Here $\mathcal{E}[\ ]$ represents the expected value or ensemble average of $[\ ]$, and $\delta(t-\tau)$ is the Dirac delta function.

The types of specific forces encountered by vehicles within this study are primarily propulsive or aerodynamic forces and are normally under some control by the vehicle. Variations in the specific force accelerations, thus, will be due to variations in the control implementation from the nominal prescribed values. If the specific forces are also dependent upon the state of the vehicle, additional variations will be realized due to the random perturbations of the position and velocity. With a caution similar to that discussed above, we assume the control implementation variations to be a linear function of independent white noise elements. Hence we express the total variation in $\underline{a}_{sf}$ as

$$\delta \underline{a}_{sf} = \frac{\partial \underline{a}_{sf}}{\partial \underline{v}} \delta \underline{v} + \frac{\partial \underline{a}_{sf}}{\partial \underline{r}} \delta \underline{r} + G_f \underline{u}_f$$

or

$$\delta \underline{a}_{sf} = F_v \delta \underline{v} + F_r \delta \underline{r} + G_f \underline{u}_f \qquad (2.15)$$

where $\mathcal{E} [\underline{u}_f(t)] = \underline{0}$

and

$$\mathcal{E} [\underline{u}_f(t) \, \underline{u}_f(\tau)^T] = Q_f(t) \, \delta (t-\tau) \qquad (2.16)$$

The linearized perturbations of the actual system may then be expressed as

$$\delta \dot{\underline{v}} = F_v \delta \underline{v} + (F_r + \Gamma) \delta \underline{r} + \underline{u}_g + G_f \underline{u}_f$$

$$\delta \dot{\underline{r}} = \delta \underline{v} \qquad (2.17)$$

Perturbations in our estimate of the position and velocity may be derived in like manner as

$$\delta \dot{\bar{\underline{v}}} = \delta \underline{a}_{sf}(\underline{v}, \underline{r}, t) + \delta \underline{a}_m + \delta \bar{\underline{g}} (\bar{\underline{r}})$$

$$\delta \dot{\bar{\underline{r}}} = \delta \bar{\underline{v}} \qquad (2.18)$$

where $\delta \underline{a}_{sf}$ and $\delta \underline{a}_m$ are expressed by (2.15) and (2.4) and where

$$\delta \bar{\underline{g}} = \Gamma \delta \bar{\underline{r}}$$

We note the absence of the white noise in the gravitational acceleration, $\underline{u}_g$, here due to the deterministic computation of $\underline{g}$ by the navigation system. Hence, we obtain

$$\delta \dot{\underline{\bar{v}}} = F_v \delta \underline{v} + F_r \delta \underline{r} + \Gamma \delta \underline{\bar{r}} + G_f \underline{u}_f + \delta \underline{a}_m$$

$$\delta \dot{\underline{\bar{r}}} = \delta \underline{\bar{v}} \tag{2.19}$$

The <u>errors</u> in estimation of the actual position and velocity may be expressed as the differences

$$\underline{e}_v = \delta \underline{v} - \delta \underline{\bar{v}}$$

$$\underline{e}_r = \delta \underline{r} - \delta \underline{\bar{r}}$$

whose derivatives may be obtained from (2.7), (2.17), and (2.19) as

$$\dot{\underline{e}}_v = \Gamma \underline{e}_r + \underline{u}_g - E \underline{b}$$

$$\dot{\underline{e}}_r = \underline{e}_v \tag{2.20}$$

In terms of the total estimation error, $\underline{e}$, defined as

$$\underline{e} = \left\{ \begin{array}{c} \underline{e}_v \\ \hline \underline{e}_r \end{array} \right\}$$

these equations become

$$\dot{\underline{e}} = B \underline{e} + E' \underline{b} + \bar{G} \underline{u}_g \tag{2.21}$$

where the matrices B, E', and $\overline{G}$ are defined as

$$B = \left[\begin{array}{c|c} O & \Gamma \\ \hline I & O \end{array}\right], \quad E' = \left[\begin{array}{c} -E \\ \hline O \end{array}\right], \quad \overline{G} = \left[\begin{array}{c} I \\ \hline O \end{array}\right]$$

and where O and I are the null and identity matrices, respectively.

We assume the mean value of the random vector, $\underline{e}$, to be zero and define the covariance matrix of $\underline{e}$ to be

$$\overline{P}(t) = \mathcal{E}\left[\underline{e}(t)\,\underline{e}(t)^T\right] \qquad (2.22)$$

We now have two sets of linear differential equations (2.9) and (2.21) which define the random errors associated with a deterministic navigation system. Through combining these two sets into one set, we may derive the differential equation for the statistical parameters of these errors as follows:

Define the augmented vector

$$\underline{\varepsilon} = \left\{\begin{array}{c} \underline{e} \\ \hline \underline{b} \end{array}\right\}$$

and its correlation matrix

$$\widetilde{P}(t) = \mathcal{E}\left[\varepsilon(t)\,\varepsilon(t)^T\right] \qquad (2.23)$$

From (2.9) and (2.21), we obtain

$$\dot{\underline{\varepsilon}} = \left[\begin{array}{c|c} B & E' \\ \hline O & D_o \end{array}\right] \underline{\varepsilon} + \left[\begin{array}{c} \overline{G} \\ \hline O \end{array}\right] \underline{u}_g \qquad (2.24)$$

from which follows

$$\dot{\tilde{P}}(t) = \mathcal{E}[\underline{\dot{\epsilon}}\,\underline{\epsilon}^T + \underline{\epsilon}\,\underline{\dot{\epsilon}}^T]$$

$$= \left[\begin{array}{c|c} B & E' \\ \hline O & D_o \end{array}\right] \tilde{P} + \tilde{P} \left[\begin{array}{c|c} B^T & O \\ \hline E'^T & D_o^T \end{array}\right]$$

$$+ \left[\begin{array}{c} \bar{G} \\ \hline O \end{array}\right] Q_g \left[\begin{array}{c|c} \bar{G}^T & O \end{array}\right] \qquad (2.25)$$

Noting that

$$\tilde{P}(t) = \left\{\begin{array}{c|c} \bar{P}(t) & P_{eb}(t) \\ \hline P_{eb}^T(t) & P_b(t) \end{array}\right\}$$

where $$P_{eb}(t) = \mathcal{E}[\underline{e}(t)\,\underline{b}(t)^T], \qquad (2.26)$$

we may write the differential equations for the individual covariance matrices, $\bar{P}$, P_{eb}, and P_b, as

$$\dot{\bar{P}}(t) = B\bar{P} + \bar{P}B^T + \bar{G}\,Q_g\,\bar{G}^T$$

$$+ E'P_{eb}^T + P_{eb}E'^T \qquad (2.27)$$

$$\dot{P}_{eb}(t) = B\,P_{eb} + P_{eb}D_o^T + E'P_b \qquad (2.28)$$

$$\dot{P}_b(t) = D_oP_b + P_bD_o^T \qquad (2.29)$$

These equations have a one-way coupling only so they may be solved in cascade, i.e., $P_b(t)$ may be found from (2.29) and used as input to find $\overline{P}_{eb}(t)$ from (2. 28), which in turn is used as input to find $\overline{P}(t)$ from (2. 27). We note that the initial value of P_b describes the statistical properties of the gyro misalignment and of the gyro accelerometer error coefficients.

2. 3 Statistical Navigation with IMU Measurements

In the deterministic navigation scheme described above, it was necessary to have a knowledge of the gravitational forces acting on the vehicle. The use of this knowledge, together with the measurements of the specific force accelerations permitted the estimation of position and velocity through integration from known initial conditions.

The error analysis of this scheme, however, required more information than was required in the navigation equations alone. In particular, we assumed that the perturbations in acceleration can be approximated by white noise having zero mean and known correlation, and that the measurement errors are random constants. If these assumtions are valid for any particular mission and reasonable statistical data is available, then we are able to obtain a good estimation of the navigation errors for that mission.

We now pose the obvious question: If such assumptions are valid and we have available the statistical data for a particular mission, is it possible to incorporate this additional information directly into the navigation system in order to reduce the inherent error in the system? The answer to this question is yes. The method of utilizing the additional information in an optimal manner is the subject of statistical estimation theory.

In practice, the use of statistical estimation theory in navigation systems requires that the navigator have available both an actual model (usually nonlinear) of the dynamics of the vehicle and a linear model describing first order perturbations about the nominal path. The position and velocity of the vehicle are determined as the sum of the nominal values (obtained from the nonlinear model) and the best estimate of the perturbations from nominal. The best estimate of the perturbations are, in turn, obtained through optimally filtering the measurements with the use of the known statistical properties of the measurement and acceleration random errors.

The design of the filter is discussed in the next section assuming perfect measurements. Later it will be shown how the design can be modified to account for random measurement errors.

2.3.1 Optimal Filter for Perfect Measurements

In order to provide the background for optimal estimation of systems having perfect measurements, we first review the results of Kalman[2] for noisy measurements. We assume the dynamics of our linear model to be described by the random process

$$\dot{\underline{x}}(t) = F\,\underline{x} + G\,\underline{u} \tag{2.30}$$

where $\underline{x}(t)$ is a vector of random variables describing the state of the system with initial conditions

$$\mathcal{E}\,[\,\underline{x}(t_o)\,] = \underline{0}$$

and $$\mathcal{E}\,[\,\underline{x}(t_o)\,\underline{x}(t_o)^T\,] = P\,(t_o)$$

and $\underline{u}(t)$ is assumed to be independent white noise with mean

$$\mathcal{E}[\underline{u}(t)] = \underline{0}$$

and covariance

$$\mathcal{E}[\underline{u}(t)\,\underline{u}(\tau)^T] = Q(t)\,\delta(t-\tau) \tag{2.31}$$

where Q(t) is nonsingular.

The state of the system is observed through measurements, $\underline{z}(t)$, which are related to the state by the linear function

$$\underline{z}(t) = H(t)\,\underline{x}(t) + \underline{\eta}(t) \tag{2.32}$$

where $\underline{\eta}(t)$ is assumed white noise in the measurements with zero mean and covariance

$$\mathcal{E}[\underline{\eta}(t)\,\underline{\eta}(\tau)^T] = R(t)\,\delta(t-\tau) \tag{2.33}$$

We assume that there is correlation between the process noise, $\underline{u}(t)$, and the measurement noise, $\underline{\eta}(t)$, and that it is adequately described by the correlation matrix

$$\mathcal{E}[\underline{u}(t)\,\underline{\eta}(\tau)^T] = S(t)\,\delta(t-\tau) \tag{2.34}$$

We can determine an estimate of $\underline{x}(t)$ through solution of the differential equation

$$\dot{\underline{\tilde{x}}} = F\underline{\tilde{x}} + K(\underline{z} - H\underline{\tilde{x}}) \tag{2.35}$$

where K is a gain matrix which "weights" the difference between the actual and predicted measurements. Kalman[2] shows that the optimum gains are obtained as

$$K(t) = (P H^T + G S) R^{-1} \tag{2.36}$$

where P(t) is the covariance of the error in the estimate of the state $\underline{x}$

$$P(t) = \mathcal{E}\left[(\underline{x}(t) - \underline{\tilde{x}}(t)) (\underline{x}(t) - \underline{\tilde{x}}(t))^T \right]$$

and is found through solution of the differential equation

$$\dot{P}(t) = F P + P F^T + G Q G^T - K R K^T \tag{2.37}$$

For a perfect measurement system, $\underline{\eta}(t) = \underline{0}$ in (2.32). The filter K(t) for this case is undefined since the matrix R is singular and cannot be inverted. Physically, such an assumption means that some linear combinations of the state variables are known exactly as soon as the measurements become available. Hence, it is possible to reduce the number of state variables to be estimated to only those variables which are not perfectly inferred from the measurements. Bryson and Johansen[13] have developed a method of reducing the state which we briefly review here.

Suppose we have measurements $\underline{y}$ related to the state through the relation

$$\underline{y} = C \underline{x} \tag{2.38}$$

Since $\underline{y}$ contains no white noise, it is reasonable to differentiate $\underline{y}$, repeatedly if necessary, using the state equations (2.30) to eliminate the appearance of $\dot{\underline{x}}$, until a new signal is obtained which contains independent elements of the white process noise, $\underline{u}(t)$. These final signals which contain white noise are grouped into a vector $\underline{z}(t)$; the elements of $\underline{y}$ and its intermediate time derivatives which are free of white noise are combined to form a vector $\underline{x}_2$. The $\underline{x}_2$ vector can be treated as a linear function of the state $\underline{x}$ which is known exactly from the measurements. Thus, we can obtain

$$\underline{x}_2 = M_2 \underline{x} \tag{2.39}$$

By defining a complementary vector

$$\underline{x}_1 = M_1 \underline{x} \tag{2.40}$$

such that $M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ is non-singular, we can infer the state $\underline{x}$ from $\underline{x}_1$ and $\underline{x}_2$ as

$$\underline{x} = M^{-1} \left\{ \begin{array}{c} x_1 \\ \hdashline x_2 \end{array} \right\} \tag{2.41}$$

The vector $\underline{z}$ can be treated as a new measurement with white noise defined as

$$\underline{z} = H_1 \underline{x} + D \underline{u} \tag{2.42}$$

With this new formulation, we can now apply the Kalman filter to estimate $\underline{x}_1(t)$ as

$$\dot{\tilde{\underline{x}}}_1 = F_{11} \tilde{\underline{x}}_1 + F_{12} \underline{x}_2 + K(\underline{z}' - H \tilde{\underline{x}}_1) \tag{2.43}$$

where the matrices F_{11}, F_{12}, and H are found as

$$\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} = \dot{M}\, M^{-1} + M\, F\, M^{-1} \tag{2.44}$$

$$H = H_1\, M^{-1} \begin{bmatrix} I \\ O \end{bmatrix} \tag{2.45}$$

and where $\underline{z}' = \underline{z} - H_1\, M^{-1} \begin{bmatrix} O \\ I \end{bmatrix} \underline{x}_2$

The optimal gain is determined as

$$K = (P_1\, H^T + G_1\, S)\, R^{-1} \tag{2.46}$$

where

$$G_1 = M_1\, G \tag{2.47}$$

$$S = Q\, D^T \tag{2.48}$$

$$R = D\, Q\, D^T \tag{2.49}$$

and where P_1 is the covariance of the error in estimate of $\underline{x}_1$

$$P_1(t) = \mathcal{E}\, [\, \underline{x}_1 - \tilde{\underline{x}}_1)\, (\underline{x}_1 - \tilde{\underline{x}}_1)^T\,]$$

and is determined through the differential equation

$$\dot{P}_1 = F_{11}\, P_1 + P_1\, F_{11}^T + G_1 Q G_1^T - K R K^T \tag{2.50}$$

The initial conditions of P_1 are obtained as

$$P_1(t_o+) = M_1(t_o)\, P(t_o+)\, M_1(t_o)^T \tag{2.51}$$

$$\tilde{\underline{x}}_1(t_o+) = M_1(t_o)\, P(t_o)\, M_2^T(t_o)$$

$$[\, M_2(t_o)\, P(t_o)\, M_2(t_o)^T\,]^{-1}\, \underline{x}_2(t_o) \tag{2.52}$$

where

$$P(t_o+) = P(t_o) - P(t_o)\, M_2^T(t_o)\, [\, M_2(t_o)$$

$$P(t_o)\, M_2^T(t_o)\,]^{-1}\, M_2(t_o)\, P(t_o) \tag{2.53}$$

and where $P(t_o)$ is the initial covariance of the error in $\underline{x}(t)$ before the first measurement has been taken.

Hence, from the estimate of $\underline{x}_1$ and the direct computation of $\underline{x}_2$, we are able to obtain the best estimate of $\underline{x}(t)$ from

$$\tilde{\underline{x}}(t) = M^{-1}(t) \begin{Bmatrix} \tilde{\underline{x}}_1(t) \\ \underline{x}_2(t) \end{Bmatrix} \tag{2.54}$$

with the covariance of error in this estimate as

$$P(t) = M^{-1}(t) \begin{bmatrix} P_1(t) & O \\ O & O \end{bmatrix} M^{-1}(t)^T \tag{2.55}$$

The above relations have been extracted from Bryson and Johansen[13] as a special case of their general treatment of colored noise in the measurements.

Since the information concerning error in the estimate of the state is obtained from the covariance matrix, $P_1(t)$, let us investigate equation (2.50) in light of the sources of this error. The first two terms on the right hand side of (2.50) relate a linear dependence of the errors at time t to the initial errors at time t_0 and upon the nominal path through the matrix F_{11}. The final term tends to reduce the estimation errors through the processing of measurements with K, while the $G_1 Q G_1^T$ term is a positive semi-definite forcing function which will always tend to increase the estimation errors due to the presence of the process noise.

We note, however, that our newly defined measurements (2.42) are driven by a linear combination of the same process noise as is driving the system (2.30) itself. Hence, there should be some reduction of the forcing term in (2.50) due to the fact that we obtain information concerning the process noise from the measurements. This premise is observed analytically by substitution of K into equation (2.50) to obtain

$$\begin{aligned}\dot{P}_1 &= F_{11}P_1 + P_1F_{11}^T + G_1QG_1^T \\ &\quad - (P_1H^T + G_1S)\,R^{-1}\,(P_1H^T + G_1S)^T \\ &= (F_{11} - G_1SR^{-1}H)\,P_1 + P_1(F_{11} - G_1SR^{-1}H)^T \\ &\quad - P_1H^TR^{-1}HP_1 + G_1QG_1^T - G_1SR^{-1}S^TG_1^T \qquad (2.56)\end{aligned}$$

From (2.48) and (2.49) we obtain

$$S\,R^{-1} = Q\,D^T\,(D\,Q\,D^T)^{-1} \qquad (2.57)$$

and

$$S\,R^{-1}\,S^T = Q\,D^T\,(D\,Q\,D^T)^{-1}\,D\,Q \qquad (2.58)$$

so that the total forcing term in $\dot{P}_1$ becomes

$$
\begin{aligned}
& G_1 (Q - S R^{-1} S^T) G_1^T \\
& \quad = G_1 (Q - Q D^T (D Q D^T)^{-1} D Q) G_1^T \\
& \quad = G_1 \overline{Q} G_1^T
\end{aligned} \tag{2.59}
$$

Since the D matrix describes the linear combination of process noise elements which is measured exactly by the measurements, $\underline{z}$, we find that this combination of noise elements is eliminated from the forcing function for $\dot{P}_1$. This elimination may be shown in a more rigorous manner if we assume the process noise vector, $\underline{u}$, to be of dimension m and the measurement vector, $\underline{z}$, of dimension r. By definition, the matrices Q and D are of rank m and r, respectively. If we note that

$$
D \overline{Q} D^T = D Q D^T - D Q D^T = O
$$

and that $\overline{Q}$ is of dimension mxm, then the rank of $\overline{Q}$ must be less than or equal to m - r. Hence, the subtraction of $S R^{-1} S^T$ from Q in (2.59) has reduced the number of independent forcing elements by at least r.

We also observe a reduction of the linear coefficients F_{11} by the product $G_1 S R^{-1} H$ in (2.56). The physical significance of this reduction can better be described when applied to the navigation system in the next section.

An interesting specialization of the optimal filter for perfect measurements is the case for which the number of independent observations in the vector $\underline{y}$ (2.38) is greater than or equal to the number of independent process noise elements in $\underline{u}$. Upon defining the new measurements, $\underline{z}$, by (2.42), we find that the D matrix will be of dimension mxm, of rank m, and hence invertible. For this special case, we obtain

$$\bar{Q} = Q - Q D^T (D^{T-1} Q^{-1} D^{-1}) D Q = O$$

$$S R^{-1} = D^{-1}$$

$$K = P_1 H^T R^{-1} + G_1 D^{-1}$$

If we define

$$\bar{H} = D^{-1} H \tag{2.60}$$

$$\underline{\bar{z}} = D^{-1} \underline{z}' = D^{-1} H \underline{x}_1 + \underline{u} = \bar{H} \underline{x}_1 + \underline{u} \tag{2.61}$$

$$\bar{F} = F_{11} - G_1 S R^{-1} H = F_{11} - G_1 \bar{H} \tag{2.62}$$

and a new gain matrix as

$$\bar{K} = P_1 \bar{H}^T Q^{-1} \tag{2.63}$$

then the estimation equation (2.43) becomes

$$\dot{\underline{\tilde{x}}}_1 = \bar{F} \underline{\tilde{x}}_1 + F_{12} \underline{x}_2 + \bar{K} (\underline{\bar{z}} - \bar{H} \underline{\tilde{x}}_1) + G_1 \underline{\bar{z}} \tag{2.64}$$

and the differential equation for P_1 (2.50) reduces to

$$\dot{P}_1 = \bar{F} P_1 + P_1 \bar{F}^T - \bar{K} Q \bar{K}^T \tag{2.65}$$

We thus find a total elimination of the positive forcing term and realize the effect of the process noise only through the reduction of the error in estimation of $\underline{x}_1$, this reduction being a function of the inverse of Q. Hence for large expected values of process noise, little gain in estimation accuracy is achieved, while if $|Q|$ is very small, then a large reduction is observed through this final term. We also note that, for an observable system the estimation errors of the optimal estimate will asymptotically approach zero.

2.3.2 Navigation with Perfect Measurements

As was suggested above, statistical navigation requires the navigator to have available a nominal model of the vehicle dynamics as well as a linear model describing random perturbations about the nominal path. Both of these models were presented in our discussion of deterministic navigation systems and are repeated here.

The nonlinear model of the vehicle equations of motion with respect to an inertial frame of reference is expressed as

$$\begin{aligned} \dot{\underline{v}}_o &= \underline{a}_{sf} + \underline{g} \\ \dot{\underline{r}}_o &= \underline{v}_o \end{aligned} \tag{2.66}$$

The random linear perturbations about this nominal path are obtained from equation (2.17) as

$$\begin{aligned} \delta\dot{\underline{v}} &= F_v\,\delta\underline{v} + (F_r + \Gamma)\,\delta\underline{r} + G_f\underline{u}_f + \underline{u}_g \\ \delta\dot{\underline{r}} &= \delta\underline{v} \end{aligned}$$

where $\quad F_v = \dfrac{\delta\underline{a}_{sf}}{\delta\underline{v}}\ , \quad F_r = \dfrac{\delta\underline{a}_{sf}}{\delta\underline{r}}\ , \quad \Gamma = \dfrac{\delta\underline{g}}{\delta\underline{r}}$

and where $\underline{u}_f$ and $\underline{u}_g$ are white noise functions driving the specific force and gravitational accelerations, respectively. We assume $\underline{u}_f$ and $\underline{u}_g$ to have zero means and covariance matrices defined as

$$\mathcal{E}\,[\,\underline{u}_f(t)\,\underline{u}_f(\tau)^T\,] = Q_f(t)\,\delta(t-\tau)$$

$$\mathcal{E}\,[\,\underline{u}_g(t)\,\underline{u}_g(\tau)^T\,] = Q_g(t)\,\delta(t-\tau)$$

We also assume that $\underline{u}_f$ and $\underline{u}_g$ are uncorrelated such that

$$\mathcal{E}\,[\,\underline{u}_f(t)\,\underline{u}_g(t)^T\,] = O$$

In general, the matrices Q_f and Q_g will be positive semi-definite at all times. For the present we will further restrict the matrix Q_f through the requirement that $G_f Q_f G_f^T$ be positive definite and hence invertible at all times.* The effect of removing this restriction will be discussed later.

If we define a vector $\underline{x}$ to represent the linear state perturbations as

$$\underline{x} = \left\{ \begin{array}{c} \delta \underline{v} \\ \hline \delta \underline{r} \end{array} \right\}$$

equation (2.17) can be written as

$$\dot{\underline{x}} = F\,\underline{x} + G\,\underline{u} \tag{2.67}$$

---

*We will show later that if this condition is satisfied, the matrix product $G_f Q_f G_f^T$ forms the measurement noise correlation matrix, R.

where

$$F = \begin{bmatrix} F_v & (F_r + \Gamma) \\ I & O \end{bmatrix} \qquad G = \begin{bmatrix} G_f & I \\ O & O \end{bmatrix}$$

and

$$\underline{u} = \begin{Bmatrix} \underline{u}_f \\ \hline \underline{u}_g \end{Bmatrix}$$

From the above definitions we can obtain the matrix Q to represent the process noise covariance as

$$Q = \begin{bmatrix} Q_f & O \\ O & Q_g \end{bmatrix}$$

Measurements are received from the IMU concerning the specific force accelerations encountered by the vehicle. Linear perturbations of these measurements with the assumption of a perfect IMU are obtained from (2.15) as

$$\underline{y} = \delta \underline{a}_{sf} = F_v \, \delta \underline{v} + F_r \, \delta \underline{r} + G_f \underline{u}_f \tag{2.68}$$

With the assumption that $G_f Q_f G_f^T$ is non-singular, the measurement vector $\underline{y}$ will contain independent white noise components without the necessity of differentiation. Hence,

$$\underline{z} = \underline{y} = H \underline{x} + D \underline{u} \tag{2.69}$$

where

$$H = \left[F_v \,\middle|\, F_r \right] \quad \text{and} \quad D = \left[G_f \,\middle|\, O \right]$$

Since there are no state variables perfectly inferred from the measurements, $\underline{x}_2$ and M_2 (in the notation of section 2.3.1) are undefined, and $M_1 = M = I$. The estimation of $\underline{x}(t)$ may be obtained from (2.43) as

$$\dot{\underline{\tilde{x}}} = F\underline{\tilde{x}} + K(\underline{z} - H\underline{\tilde{x}}) \tag{2.70}$$

where $K = (PH^T + GS)R^{-1}$

From (2.48) and (2.49), we find

$$S = QD^T = \begin{bmatrix} Q_f G_f^T \\ \hline O \end{bmatrix}$$

$$GS = \begin{bmatrix} G_f Q_f G_f^T \\ \hline O \end{bmatrix}$$

$$R = DQD^T = G_f Q_f G_f^T$$

so that

$$K = PH^T R^{-1} + \begin{bmatrix} I \\ \hline O \end{bmatrix}$$

The differential equation for P is obtained from (2.50) as

$$\dot{P} = FP + PF^T + GQG^T - KRK^T$$

From the definitions of G, Q, R, and K above, the final two terms in this equation may be obtained as

$$G Q G^T = \begin{bmatrix} (G_f Q_f G_f^T + Q_g) & O \\ O & O \end{bmatrix}$$

$$K R K^T = P H^T R^{-1} H P + P H^T [I \quad O] + \begin{bmatrix} I \\ O \end{bmatrix} H P + \begin{bmatrix} I \\ O \end{bmatrix} R [I \quad O]$$

$$= P H^T R^{-1} H P + P \begin{bmatrix} H^T & O \end{bmatrix} + \begin{bmatrix} H \\ O \end{bmatrix} P + \begin{bmatrix} G_f Q_f G_f^T & O \\ O & O \end{bmatrix}$$

Thus,

$$\dot{P} = B P + P B^T + \begin{bmatrix} Q_g & O \\ O & O \end{bmatrix} - P H^T R^{-1} H P \qquad (2.71)$$

where

$$B = F - \begin{bmatrix} H \\ O \end{bmatrix} = \begin{bmatrix} O & \Gamma \\ I & O \end{bmatrix}$$

Through the above definition of K, it is possible to reduce the estimation equation (2.70) to

$$\dot{\underline{\tilde{x}}} = B \underline{\tilde{x}} + \left\{ \frac{z}{\underline{0}} \right\} + P H^T R^{-1} (\underline{z} - H \underline{\tilde{x}}) \qquad (2.72)$$

It is now possible to formulate the statistical navigation system with the use of the nonlinear model (2.66) and the linear estimation model (2.72). The solution of equations (2.66) with known initial conditions allows the same result as is obtained through the deterministic navigation scheme (2.12). The navigation accuracy is increased, however, through simultaneous solution of equations (2.72) and (2.71) with initial conditions

$$\underline{\tilde{x}}(t_o) = \underline{0}$$

$$P(t_o) = \mathcal{E} [\underline{x}(t_o) \underline{x}(t_o)^T]$$

and with known covariance matrices Q_f and Q_g. The optimal estimate of position and velocity is then obtained as

$$\begin{Bmatrix} \tilde{\underline{v}}(t) \\ \tilde{\underline{r}}(t) \end{Bmatrix} = \begin{Bmatrix} \underline{v}_o(t) \\ \underline{r}_o(t) \end{Bmatrix} + \tilde{\underline{x}}(t)$$

The statistical properties of the error resulting from this estimation scheme are found directly through the covariance matrix

$$P(t) = \mathcal{E}\,[\tilde{\underline{e}}(t)\,\tilde{\underline{e}}(t)^T]$$

where $$\tilde{\underline{e}}(t) = \begin{Bmatrix} \underline{v}(t) \\ \underline{r}(t) \end{Bmatrix} - \begin{Bmatrix} \tilde{\underline{v}}(t) \\ \tilde{\underline{r}}(t) \end{Bmatrix}$$

Let us now consider the assumption that the product $G_f Q_f G_f^T$ remains non-singular at all times. In general, this assumption implies that the number of independent noise elements which produce random variations in the specific force accelerations is greater than or equal to the number of independent components of the specific force acceleration vector, $\underline{a}_{sf}$. Since we have assumed that the measurement system observes the same $\underline{a}_{sf}$ as employed in the nominal model (2.66), the linear measurement perturbations $\underline{y}$ defined by equation (2.68), then also contain independent white noise elements.

If the assumption is not valid at any time, i.e., if $G_f Q_f G_f^T$ becomes positive <u>semi</u>-definite, then at least one element within the measurement vector contains dependent (or zero) white noise, and hence may be considered a perfect measurement. Recognizing this fact, it is necessary to differentiate this measurement as discussed in section 2.3.1 until an independent white noise element is obtained. This element will necessarily be some linear combination of the noise vector $\underline{u}_g$.

The newly defined measurement may be expressed by the equation

$$\underline{z}' = H'\underline{x} + D'\underline{u}$$

where $D' Q D'^T$ is non-singular.

Since a linear combination of the elements of the state $\underline{x}$ is implied by the perfect measurement(s), it will be necessary to reduce the state estimation equations by the Bryson-Johansen method described in the previous section and to employ the estimation equations (2.39) thru (2.55).

It is now possible to state come general conclusions concerning the statistical inertial navigation scheme presented here with the basic assumption that the measurements $\underline{y}$ provide perfect duplication of the variation in the entire specific force acceleration vector $\underline{a}_{sf}$, i.e., that equation (2.68) is representative of the variation in nominal information received from the IMU. From (2.68), we find that the total process noise derived from the specific force accelerations $\underline{u}_f$ is perfectly inferred from the measurements. Also, we note that the $\underline{a}_{sf}$ coefficient matrices, F_v and F_r, which are included as coefficients in the linear system through the matrix F in (2.67), are exactly duplicated in the measurements (2.68). Due to the perfect inference of these quantities by the measurements, we find that the result of applying statistical estimation theory to the linear system (2.67) is to remove these quantities from the system itself and to employ them only in the formulation of the filter. This may be stated in a more formal manner through the differential equation of the estimation error covariance matrix, P, which may be written in general as

$$\dot{P} = B' P + P B'^T + G' Q_g G'^T - K' R' K'^T \qquad (2.73)$$

The linear coefficients, B', will contain only those elements derived from linear perturbations in the non-specific force accelerations. The positive forcing term, $G' Q_g G'^T$, will, in like manner, contain only linear combinations of the noise components driving the non-specific force accelerations (if the matrix product $G_f Q_f G_f^T$ is singular, then the elements of $\underline{u}_g$ accumulated through differentiation of the perfect measurement(s) will be removed from this forcing term and included in the R' matrix).

The final term $K' R' K'^T$ is a positive semi-definite matrix which will tend to reduce the mean squared estimation errors. The covariance matrix R' as well as the gains K' will contain all of the elements of Q_f plus a linear combination of the elements of Q_g obtained through differentiation of the measurements. The matrix K' will contain the coefficient matrices, F_v and F_r, as well as other terms which may be obtained from the necessity of measurement differentiation.

Another significant conclusion can be derived from study of the optimal filter gains defined by (2.36) as

$$K = (PH^T + GS) R^{-1}$$

We have already noted the effect of the cross-correlation term $G S R^{-1}$

in eliminating specific force acceleration dependent coefficients and noise components from the matrix F and the positive forcing term $G\,Q\,G^T$. The remainder of the filter gains, consisting of the product $P\,H^T\,R^{-1}$, is thus employed to reduce the magnitude of the navigation errors. The physical basis of this reduction is that the navigator is able to discern some information concerning the variables being estimated from the correlation of the measurements with variations in these variables. This correlation is found directly through the matrix H, or more basically through the partial derivative matrices F_v and F_r. Hence we may conclude that if the specific forces acting on the vehicle are in no way dependent upon the position and velocity of the vehicle, then the matrix H and, in turn, the gain K will be identically zero and no advantage is realized in the statistical filtering navigation over the deterministic navigation scheme.

2.3.3 Inclusion of Measurement Errors

Thus far, we have considered optimal statistical navigation for perfect inertial measurements. We now study the effects of including the errors associated with the IMU measurements.

In section 2.1, it was pointed out that sufficient evidence has been found, through studies of IMU systems, to validate the assumption of random constant error coefficients in the IMU over short periods of time. With this assumption, it was possible to formulate the errors in sensed acceleration by the IMU as

$$\delta \underline{a}_m = E\,\underline{b}$$

where $$\dot{\underline{b}} = D_o\,\underline{b}$$

and where the matrices E and D_o are defined by equations (2.8) and (2.9). The vector $\underline{b}$ is a random vector defined by (2.6) containing the gyro drift components and the gyro and accelerometer error coefficients. It is assumed to have a mean value of $\underline{0}$ and a covariance matrix $P_b(t)$ defined by (2.10). The effect of including these measurement errors into the linear system is seen through adding $\delta \underline{a}_m$ to the measurements $\underline{y}$ (2.68) to give

$$\underline{y} = \delta \underline{a}_{sf} + \delta \underline{a}_m = F_v \delta \underline{v} + F_r \delta \underline{r} + G_f \underline{u}_f + E \underline{b} \qquad (2.74)$$

In order to include the measurement errors in the navigation scheme, it would be necessary to include the elements of $\underline{b}$ as additional state variables to be estimated. To realize this, we would define an augmented state vector $\underline{X}$ as

$$\underline{X} = \begin{Bmatrix} \delta \underline{v} \\ \delta \underline{r} \\ \underline{b} \end{Bmatrix}$$

From (2.9) and (2.17) we obtain the differential equation for $\underline{X}$ as

$$\dot{\underline{X}} = \tilde{F} \underline{X} + \tilde{G} \underline{u}$$

where

$$\tilde{F} = \begin{bmatrix} F_v & F_r + \Gamma & O \\ I & O & O \\ O & O & D_o \end{bmatrix} \qquad \tilde{G} = \begin{bmatrix} G_f & I \\ O & O \\ O & O \end{bmatrix}$$

and $\qquad \underline{u} = \begin{Bmatrix} \underline{u}_f \\ \underline{u}_g \end{Bmatrix}$

If we assume $G_f Q_f G_f^T$ to be non-singular, the measurements $\underline{\tilde{z}}$ may be defined as

$$\underline{\tilde{x}} = \tilde{H} \underline{X} + D \underline{u} \tag{2.75}$$

where

$$\tilde{H} = \begin{bmatrix} F_v & F_r & E \end{bmatrix} \qquad D = \begin{bmatrix} G_f & O \end{bmatrix}$$

The optimal gain for this augmented system is now obtained as

$$\tilde{K} = (\tilde{P} \tilde{H}^T + \tilde{G} S) R^{-1}$$

$$= \tilde{P} \tilde{H}^T R^{-1} + \begin{bmatrix} I \\ O \\ O \end{bmatrix}$$

where $\quad S = Q D^T$

and $\quad \tilde{P} = \mathcal{E} [(\underline{X} - \underline{\tilde{X}}) (\underline{X} - \underline{\tilde{X}})^T]$

The augmented random vector $\underline{X}(t)$ could be estimated as

$$\dot{\underline{\tilde{X}}} = \tilde{F} \underline{\tilde{X}} + \tilde{K} (\underline{\tilde{z}} - \tilde{H} \underline{\tilde{X}})$$

or as

$$\dot{\underline{\tilde{X}}} = \tilde{B} \underline{\tilde{X}} + \begin{bmatrix} z \\ \underline{0} \\ \underline{0} \end{bmatrix} + \tilde{P} \tilde{H}^T R^{-1} (\underline{\tilde{z}} - \tilde{H} \underline{\tilde{X}}) \tag{2.76}$$

and the differential equation for $\tilde{P}$ would be derived as

$$\dot{\tilde{P}} = \tilde{B} \tilde{P} + \tilde{P} \tilde{B}^T - \tilde{P} \tilde{H}^T R^{-1} \tilde{H} \tilde{P} + \begin{bmatrix} Q_g & O & O \\ O & O & O \\ O & O & O \end{bmatrix} \tag{2.77}$$

where

$$\widetilde{B} = \begin{bmatrix} O & \Gamma & -E \\ I & O & O \\ O & O & D_o \end{bmatrix}, \quad R = G_f Q_f G_f^T$$

Estimation of the entire augmented vector $\underline{X}$ would indeed provide us with the optimal navigation system in the sense of maximum accuracy in estimation. Such a navigation system would be employing every available fragment of information concerning the vehicle dynamics and measurement system and the random noise components affecting them.

As noted in section 2.1, however, the $\underline{b}$ vector could contain as many as 81 elements. Adding to this the six-dimensional $\underline{x}$ vector, the navigator would conceivably be called upon to estimate up to 87 individual state variable variations. The estimate of this state would, in turn, require the simultaneous integration of a total of 7743 variables according to equations (2.66), (2.9), (2.76), and (2.77). This number could be reduced considerably due to the fact that many of the error coefficients within the $\underline{b}$ vector have an insignificant effect upon the actual IMU errors and thus could be eliminated. But, even if the size of the $\underline{b}$ vector were reduced by such elimination to half its size, the navigator would still be burdened with a large quantity of numerical integration.

It is because of this insurmountable computation time that the designer of a statistical navigation system must be satisfied with a suboptimal system, that is, one which does not utilize all of the information available and hence does not allow a true minimum error in estimation.

The size of the state vector to be estimated and employed in the navigation system will be determined through the individual designer's criteria in considering the tradeoffs between navigation accuracy and system complexity.

The errors introduced by not estimating any elements of the $\underline{b}$ vector may be found by adding the measurement error equations developed in section 2.2 for the deterministic system to the equation for covariance of the estimation errors determined for the filtering system. The covariance matrix of the actual estimation errors would then be found as the solution of the equation

$$\dot{P} = F P + P F^T + G Q G^T - K R K^T + E' P_{eb}^T + P_{eb} E'^T \tag{2.78}$$

where the matrix K is the same as employed in the estimation for perfect measurements and P_{eb} is obtained from solution of equations (2.28) and (2.29).

2.4 Comparison of Deterministic and Statistical Navigation Schemes

There are several criteria upon which we may base a comparison of the two inertial navigation schemes presented here. Among these, the most important criteria would be navigation accuracy and system practicability.

A quantitative comparison of navigation accuracy could be obtained through solution of the error covariance equations associated with each scheme with given initial conditions along a specified nominal path. The differential equation for the covariance matrix corresponding to the deterministic navigation scheme is given in equation (2.27) as

$$\dot{\bar{P}} = B\bar{P} + \bar{P}B^T + \bar{G}Q_g\bar{G}^T + E'P_{eb}{}^T + P_{eb}E'^T \tag{2.79}$$

The first two terms in this equation show the dependence of the errors upon the initial errors and upon the non-specific force variations along the path while the third term reveals the positive linear dependence upon white noise in the non-specific force accelerations. The measurement errors are added through the final two terms.

The form of the error covariance differential equation for the statistical navigation scheme will depend upon the number of state variables to be estimated and the characteristics of the process noise driving the specific force acceleration perturbations. We will assume for the present that the statistical navigator assumes perfect measurements and that the matrix product $G_f Q_f G_f{}^T$ is non-singular. Then the navigation errors for this navigator may be obtained from equations (2.78) and (2.71) as

$$\dot{P} = BP + PB^T + \begin{bmatrix} Q_g & O \\ O & O \end{bmatrix} + E'P_{eb}{}^T$$

$$+ P_{eb}E'^T - PH^TR^{-1}HP \tag{2.80}$$

We note that the first five terms in (2.80) are identical to those in (2.79) if P is replaced by $\bar{P}$. The final term in (2.80) is a positive semi-definite quantity which is subtracted from the differential equation for P. Thus, we are able to show a clear reduction of navigation errors through the statistical navigation scheme if the H matrix is not identically zero. It can also be shown through study of equation (2.77) that the only effect of estimating some of the elements of the measurement error vector $\underline{b}$ in addition to the position and velocity variations would be to increase the size of the filter term and thus achieve even better accuracy.

The practicability of the statistical navigation scheme would have to be investigated in light of both the gain in accuracy over the deterministic scheme and the availability of on-board computer time and storage capacity. As noted above, no gain in navigation accuracy can be realized if the matrix H, describing the correlation between the specific force accelerations and the parameters being estimated, is zero. Such an effect is observed in attempting to navigate under the accelerating forces of propulsive thrust. The magnitude and direction of the thrust vector are controlled through commands from the guidance system and are independent of the actual position and velocity of the vehicle. Because of this independence, it would be impossible to improve the estimate of position and velocity obtained from the deterministic inertial navigation system through the statistical estimation techniques (using acceleration measurement data only) without estimating elements of the measurement error coefficients, $\underline{b}$.

Aerodynamic forces, however, are found to depend upon both the position and relative velocity of the vehicle. Due to this dependence, the navigation accuracy would be improved with the use of statistical navigation.

Computer requirements for the statistical navigation scheme could be evaluated from consideration of the estimation equations presented in section 2.3. For estimation of n state variables, it is apparent that the solution of $n(n+1)$ differential equations is required in addition to the integration of the n nominal equations necessary in the deterministic scheme. This additional burden on the computer may increase numerical computation times to exceed the real time limit. If a predetermined nominal path is to be followed, it may be permissible to store a large portion of information corresponding to this nominal path. This would eliminate real time integration of the covariance matrix P and computation of the filter K but only through an increased requirement for storage capacity.

A more detailed comparison of the two navigation schemes is presented in the re-entry application which follows.

CHAPTER III

DEVELOPMENT OF STATISTICAL INERTIAL NAVIGATION FOR APOLLO RE-ENTRY

The final phase of the Apollo manned lunar landing mission will be the re-entry of the command module through the earth's atmosphere to a predetermined landing site. Prior to entering the atmosphere, the command module will be directed into a safe entry corridor and aligned to an aerodynamically stable attitude which allows a small lifting force on the vehicle. A limited amount of re-entry path control is achieved by rolling the vehicle to change the direction of this lift force. Upon entering the atmosphere, the guidance system must steer the vehicle to the desired landing point with high accuracy while avoiding excessive accelerations and possible skip out with supercircular velocity. If the prescribed landing conditions require a long range to be achieved by the vehicle after entry, it will usually be necessary to perform a controlled skip out of the atmosphere through a sub-orbital ballistic flight as shown in Figure 3.1. Lickly, et. al.[21] show that the errors in terminal range are highly sensitive to errors in guiding the vehicle during initialization of this skip maneuver.

Guidance accuracy can be no greater than the accuracy of the navigation information supplied to the guidance system. With initial conditions of the vehicle position and velocity obtained from the mid-course navigator prior to re-entry, the acceleration data obtained

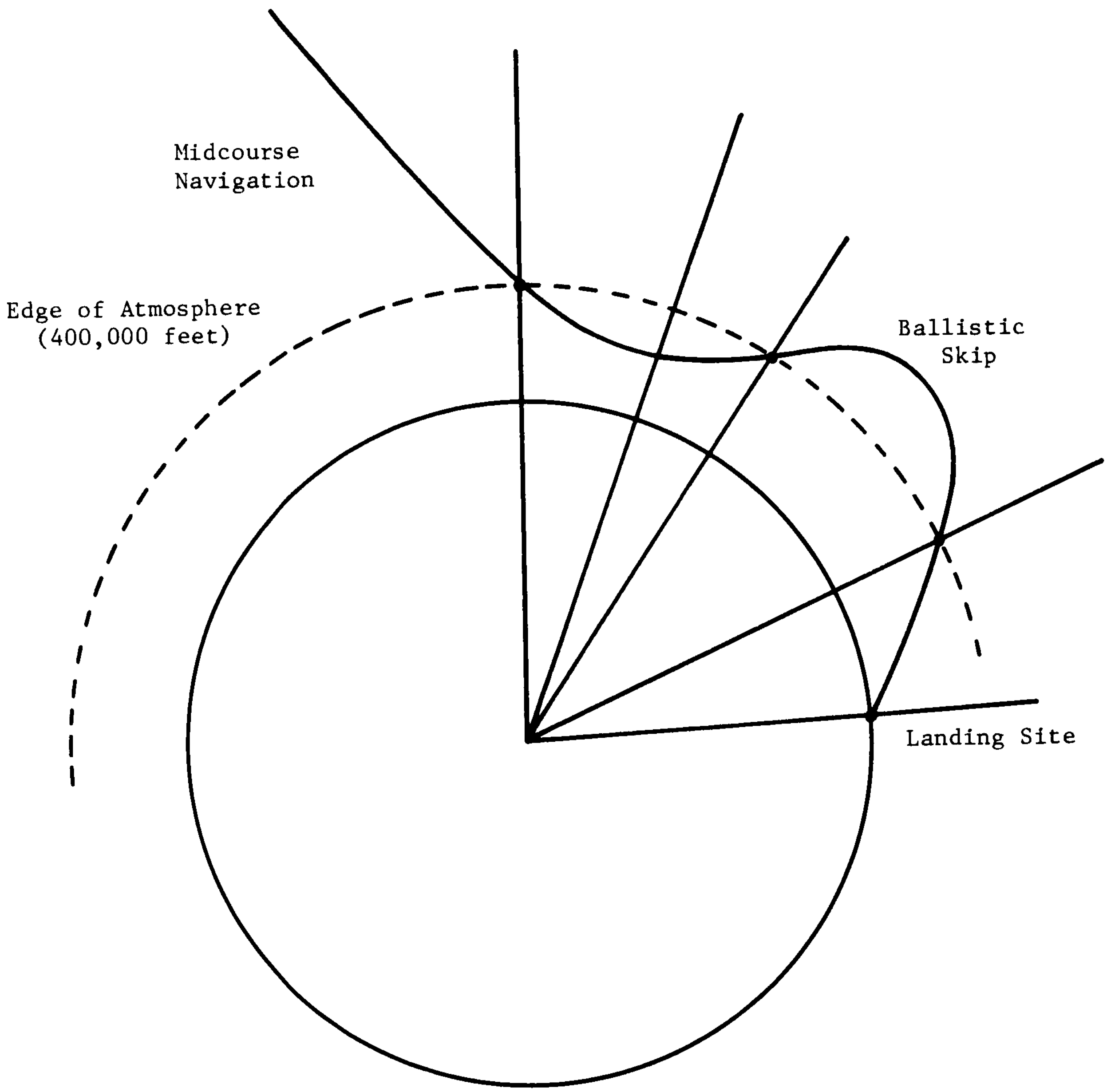


Figure 3.1

Typical Reentry Trajectory

from an IMU during re-entry could be used to determine the present state of the vehicle. With the conventional deterministic integration of these accelerations, any uncertainties in the initial values received from the midcourse system will be carried through the entire re-entry phase. Since the aerodynamic forces acting on the vehicle during re-entry are known to depend on the velocity and the atmospheric density, it would be possible to reduce the effects of these initial condition errors with the incorporation of statistical estimation theory into the navigation scheme.

This chapter will employ the statistical estimation theory developed in Chapter II in the design of an inertial navigation system for the re-entry portion of the Apollo mission. The three dimensional dynamics of the Apollo command module will be presented and subsequently simplified to two dimensional motion in an equatorial plane about a rotating earth. Random variations about this nominal path will then be used with appropriate random process noise to define the optimal statistical filter for the assumed two dimensional model.

3.1 Description of Apollo Re-entry Vehicle

The Apollo command module is a conical capsule as shown in Figure 3.2. Its shape and center of gravity provide an aerodynamic trim orientation with the heat shield forward and one side nearly parallel to the wind direction. Control jets are provided to damp out oscillations in pitch and yaw and thus maintain stability in this trim attitude. The angle of attack provided by this orientation is 22° and creates a ratio of lift to drag of 0.3. Roll of the vehicle about the

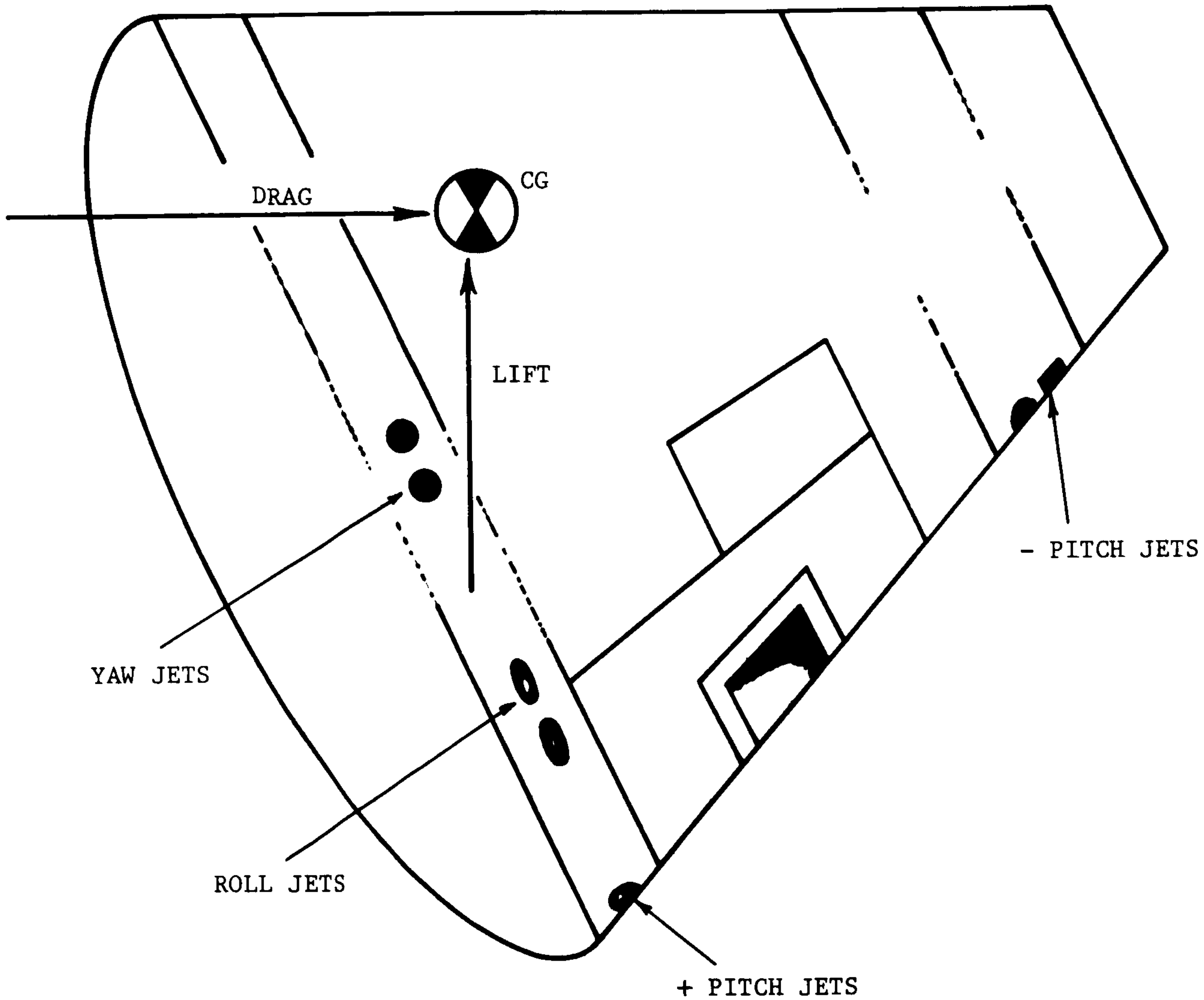


Figure 3.2

Apollo Command Module

velocity direction is controlled by on-off operation of hypergolic thrusters to allow for desired orientation of the lift force.

3.2 Vehicle Dynamics

With the assumption that the motion of the vehicle may be adequately described as that of point mass about a spherical rotating earth, we may write the vehicle equations of motion in the rotating spherical coordinate frame shown in Figure 3.3 as

$$\dot{V} = f_V - \frac{g_o R_e^2 \sin\gamma}{(R_e + h)^2} + (R_e + h)\,\Omega^2 \cos\lambda\,(\sin\gamma\cos\lambda - \cos\gamma\sin\psi\sin\lambda)$$

$$V\dot{\gamma} = f_\gamma + \frac{V^2 \cos\gamma}{(R_e + h)} - \frac{g_o R_e^2 \cos\gamma}{(R_e + h)^2} + 2\Omega V\cos\psi\cos\lambda$$

$$+ (R_e + h)\,\Omega^2\cos\lambda\,(\cos\gamma\cos\lambda + \sin\gamma\sin\psi\sin\lambda)$$

$$V\cos\gamma\,\dot{\psi} = f_\psi - \frac{V^2}{(R_e + h)}\cos^2\gamma\cos\psi\tan\lambda$$

$$+ 2\Omega V(\sin\psi\sin\gamma\cos\lambda - \sin\lambda\cos\gamma)$$

$$- (R_e + h)\,\Omega^2\cos\lambda\cos\psi\sin\lambda$$

$$\dot{h} = V\sin\gamma \tag{3.1}$$

$$\dot{\lambda} = \frac{V\cos\gamma\sin\psi}{(R_e + h)}$$

$$\dot{\theta} = \frac{V\cos\gamma\cos\psi}{(R_e + h)\cos\lambda}$$

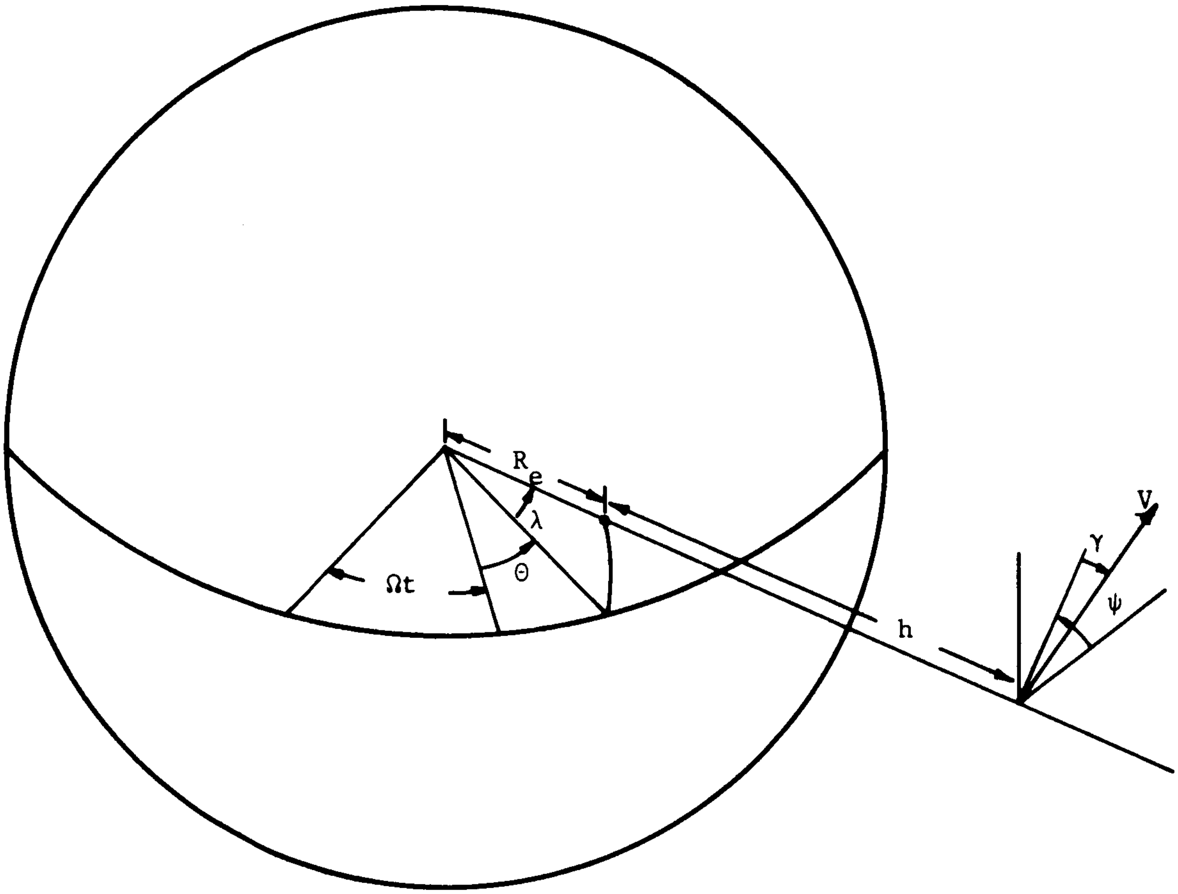


Figure 3.3

Rotating Coordinate System

where

V = velocity relative to rotating earth

γ = flight path angle measured positive above the horizon

ψ = azimuth angle

h = altitude

λ = latitude

θ = longitude

R_e = radius of earth

g_o = gravitational acceleration at surface of earth

Ω = earth rotation rate

and where f_V, f_γ, and f_ψ are the aerodynamic specific forces acting on the vehicle. In terms of the roll angle, ϕ, and the sideslip angle, ζ, these forces may be defined as

$$f_V = \frac{\rho A_c V_a^2}{2m} (C_Y \sin\zeta - C_D \cos\zeta) \tag{3.2}$$

$$f_\gamma = \frac{\rho A_c V_a^2}{2m} (C_L \cos\phi - C_D \sin\phi \sin\zeta - C_Y \sin\phi \cos\zeta)$$

$$f_\psi = -\frac{\rho A_c V_a^2}{2m} (C_D \cos\phi \sin\zeta + C_Y \cos\phi \cos\zeta + C_L \sin\phi)$$

where

ρ = atmospheric density

A_c = cross-sectional area of vehicle

V_a = relative velocity of vehicle with respect to the surrounding air

and where C_L, C_D, and C_Y are, respectively, the lift, drag, and yaw coefficients.

In order to maintain a reasonably simplified analysis of the navigation system, we will assume that the vehicle moves in an equatorial plane about the earth and neglect all motions of the vehicle out of this plane. With this assumption, we obtain

$$\dot{\psi} = \psi = 0$$

and $$\dot{\lambda} = \lambda = 0$$

and the two dimensional equations of motion reduce to

$$\dot{V} = f_v - \frac{g_o R_e^2 \sin\gamma}{(R_e + h)^2} + (R_e + h)\,\Omega^2 \sin\gamma$$

$$\dot{\gamma} = \frac{f_\gamma}{V} + \frac{V\cos\gamma}{(R_e + h)} - \frac{g_o R_e^2 \cos\gamma}{V\,(R_e + h)^2} + 2\Omega + \frac{(R_e + h)\,\Omega^2 \cos\gamma}{V}$$

$$\dot{h} = V \sin\gamma$$

$$\dot{\theta} = \frac{V\cos\gamma}{(R_e + h)}$$

The lift, drag, and yaw coefficients of the vehicle will, in general, be nonlinear functions of the angle of attack, α, the sideslip angle, ζ, the Mach number, the Reynolds number, and the angular rates of the vehicle. At the high supersonic velocities encountered during re-entry, however, these coefficients may be assumed dependent only on the angle of attack and the sideslip angle. We thus approximate the aerodynamic coefficients by polynomial functions of α and ζ as

$$C_L = C_{Lo}\alpha + C_{L1}\alpha^3$$
$$C_D = C_{Do} + C_{D1}\alpha^2 + C_{D2}\zeta^2 \qquad (3.3)$$
$$C_Y = C_{Yo}\zeta + C_{Y1}\zeta^3$$

If the design aerodynamic trim attitude is maintained, then $\zeta = 0$, $\alpha = 22^o$, $C_Y = 0$, and the nominal specific forces f_v and f_γ may be written as

$$f_v = -\frac{\rho A_c V_a^2 C_D}{2m} \qquad f_\gamma = \frac{\rho A_c V_a^2 C_L \cos\phi}{2m} \qquad (3.4)$$

Without knowledge of the wind conditions at the altitudes to be considered (above 100,000 feet), we neglect them and assume that the atmosphere rotates with the earth so that

$$V_a = V$$

We will also approximate the atmospheric density as an exponential function of altitude as

$$\rho = \rho_o e^{-\beta h} \qquad (3.5)$$

where ρ_o and β are constants.

With these considerations, the equations of motion become

$$\dot{V} = -\frac{\rho_o e^{-\beta h} A_c C_D V^2}{2m} - \frac{g_o R_e^2 \sin\gamma}{(R_e + h)^2} + (R_e + h)\,\Omega^2 \sin\gamma$$

$$\dot{\gamma} = \frac{\rho_o e^{-\beta h} A_c C_L V \cos\phi}{2m} + \frac{V\cos\gamma}{(R_e + h)} - \frac{g_o R_e^2 \cos\gamma}{V\,(R_e + h)^2}$$
$$+ 2\Omega + \frac{(R_e + h)\,\Omega^2 \cos\gamma}{V} \qquad (3.6)$$

$$\dot{h} = V \sin \gamma$$

$$\dot{\theta} = \frac{V \cos \gamma}{(R_e + h)}$$

3.3 Random Error Sources

Errors in navigating with this model would be derived from the assumptions intentionally employed to reduce the complexity of the model, from uncertainties in the constant parameters and initial conditions, and from random disturbances which could be described only through their statistical characteristics. In this study, we neglect deterministic variations which could be extracted from a more sophisticated model and consider only the variations due to unpredictable random disturbances.

The effectiveness of any statistical estimation scheme is based upon the ability to describe the statistical properties of random disturbances affecting the physical process and the measurements. In the present study we have assumed the inertial measurement system to contain no high frequency random noise. Therefore the random disturbances in the measurements arise solely from the process noise affecting the vehicle dynamics. If this process noise can be modeled with sufficient white noise to produce independent white noise elements in the measurements, then the Kalman filtering theory may be employed directly in describing an optimal filter. If, on the other hand, the specific force accelerations are not perturbed by independent white noise elements, then recognition must be made of the fact that some perfect knowledge of the state of the vehicle is obtained from the measurements. In this case, it was found in Chapter II that the statistical filter may be defined for a reduced system, possibly utilizing derivatives of the perfect measurements.

Since the elements of process noise play such an important role in the design of the statistical filter, careful consideration should be afforded the modeling of their statistical properties.

The primary random disturbances affecting the aerodynamic forces during re-entry could be attributed to

1. Variations in the atmospheric density.
2. Random winds.
3. Unsteady motions of the vehicle about the aerodynamic trim attitude.
4. Effects of control implementation errors.
5. Disturbances in the aerodynamic forces due to unsteady flow around the vehicle and to the effects of mass ablation from the heat shield.

Since no supercircular re-entry flights of an Apollo type vehicle have been made, a statistical analysis of these disturbances must be conjectured from available ground test data and intuitive reasoning. Each of the above disturbances are considered both qualitatively and quantitatively. The quantitative comparison is obtained through examination of the effect of each disturbance on the variations in the aerodynamic force accelerations along the nominal re-entry trajectory chosen for analysis in Chapter IV.

3.3.1 Atmospheric Density Variations

Some studies of atmospheric density variations have been made with limited data and theoretical model approximations. Reference (22) shows these variations to depend on season, latitude, and altitude. Cole and Kantor[23] show approximate extreme values of the variations between 30 and 90 kilometers as obtained through the assumption that the atmosphere remains in hydrostatic equilibrium. The maximum values of daily variations in density (with 95 per cent certainty) at a latitude of 15 degrees north are shown by this study to depend on altitude and are represented as percentage of nominal density in Figure 3.4 on page 60.

From these findings, we could attempt a crude model of the statistical characteristics of the density variations. We first assume that any density perturbation from standard would be highly correlated in time so that at a given position in the upper atmosphere it could be considered as a random constant. We would expect different values of the random variations as we change altitude, however. Since the uncertainty in density would not change rapidly with altitude, we could not assume it to be white noise, but could possibly construct a shaping filter to represent the correlation with altitude as

$$\frac{d}{|dh|} \delta\rho = -\frac{1}{h_\rho} \delta\rho + \frac{1}{h_\rho} w_\rho (h) \qquad (3.7)$$

where $\delta\rho$ is the uncertainty in density, h_ρ is the correlation altitude and where $w_\rho(h)$ is white noise in altitude with

$$\mathcal{E} [w_\rho(h)] = 0$$

$$\mathcal{E} [w_\rho(h) \, w_\rho(h')] = q'_\rho (h) \, \delta (h-h')$$

The absolute value in (3.7) is necessary to insure stability of the shaping filter.

Since the model of the vehicle dynamics employs time as the independent variable, it will be necessary to transform the density variation model to one which is time dependent. This can be done by multiplying equation (3.7) by the altitude rate $\dot{h}$ to obtain

$$\delta\dot{\rho} = |\dot{h}| \frac{d}{|dh|} \delta\rho = -\frac{|\dot{h}|}{h_\rho} \delta\rho + \frac{|\dot{h}|}{h_\rho} w_\rho(h) \qquad (3.9)$$

With a linearity assumption, we may also obtain

$$\delta(t-\tau) = |\dot{h}| \, \delta(h-h')$$

Then treating the entire forcing function $\frac{|\dot{h}|}{h_\rho} w_\rho(h)$ as white noise in time, $u_\rho(t)$, we obtain

$$\delta\dot{\rho} = -\frac{|\dot{h}|}{h_\rho} \delta\rho + u_\rho(t)$$

where

$$\mathcal{E}[u_\rho(t)] = 0$$

$$\mathcal{E}[u_\rho(t)\, u_\rho(\tau)] = q_\rho(t)\, \delta(t-\tau)$$

With essentially no knowledge of the propagation of density perturbations at altitudes above 100,000 feet, we will assume that the correlation altitude, h_ρ, may be represented as the inverse of the constant β defined by (3.5), to obtain

$$\delta\dot{\rho} = -\beta\, |\dot{h}|\, \delta\rho + u_\rho \qquad (3.10)$$

The forced (or particular) solution to (3.10) may be written as

$$\delta\rho(t) = \int_{-\infty}^{t} e^{-\beta|\dot{h}|(t-\tau)} u_\rho(\tau)\, d\tau$$

and hence the variance of $\delta\rho$ as

$$\sigma_\rho^2 = \overline{\delta\rho(t)^2} = \int_{-\infty}^{t} d\tau_1 \int_{-\infty}^{t} d\tau_2\, e^{-\beta|\dot{h}|(t-\tau_1)} e^{-\beta|\dot{h}|(t-\tau_2)} q_\rho(\tau_1)\delta(\tau_2-\tau_1)$$

On a quasi-stationary basis, we treat $\dot{h}$ and q_ρ as constants to allow integration of this function. Hence

$$\sigma_\rho^2 = \int_{0}^{\infty} e^{-2\beta|\dot{h}|(t-\tau_1)} q_\rho\, d\tau_1$$

Then letting $\tau = t-\tau_1$ and $d\tau = -d\tau_1$, we obtain

$$\sigma_\rho^2 = \int_{0}^{\infty} e^{-2\beta|\dot{h}|\tau} q_\rho\, d\tau = \frac{q_\rho}{2\beta|\dot{h}|} e^{-2\beta|\dot{h}|\tau} \Bigg|_{0}^{\infty}$$

$$\sigma_\rho^2 = \frac{q_\rho}{2\beta|\dot{h}|}$$

With these simplifying assumptions, we can now express q_ρ as

$$q_\rho = 2\beta|\dot{h}|\sigma_\rho^2$$

According to Reference (23) the standard deviation σ_ρ of the density variation could be represented as a percentage of nominal density as

$$\sigma_\rho(h) = \sigma'_\rho(h)\,\rho$$

where $\sigma'_\rho(h)$ is represented by a curve such as shown in Figure 3.4. For simplification we assume $\sigma'_\rho(h)$ to be a linear function of altitude so that

$$\sigma_\rho(h) = (k_1 + k_2 h)\,\rho \tag{3.11}$$

where k_1 and k_2 are constants with values

$$k_1 = \frac{.2}{3}$$

$$k_2 = \frac{10^{-6}}{3}\ \mathrm{ft}^{-1}$$

to give the linear curve shown in Figure 3.4. The result of integrating the variance of $\delta\rho$ along a nominal trajectory from an initial RMS uncertainty $\delta\rho(0)$ of 2×10^{-11} slug/ft^3 (or approximately 0.2ρ) is also shown in Figure 3.4. The predominant increase in $\frac{\delta\rho}{\rho}$ occurs during the skip maneuver while $\dot{h}$ is positive. The results obtained would suggest that the model is not truly representative of the atmospheric density variations. However, since the intent here is not to develop a highly sophisticated model but rather to illustrate the method of including such a model in the statistical navigation system, the above model will be accepted as adequate.

The effect of a small variation in density, $\delta\rho$, on the aerodynamic force accelerations may be evaluated as

$$\delta f_V = \frac{f_V}{\rho}\ \delta\rho$$

$$\delta f_\gamma = \frac{f_\gamma}{\rho}\ \delta\rho$$

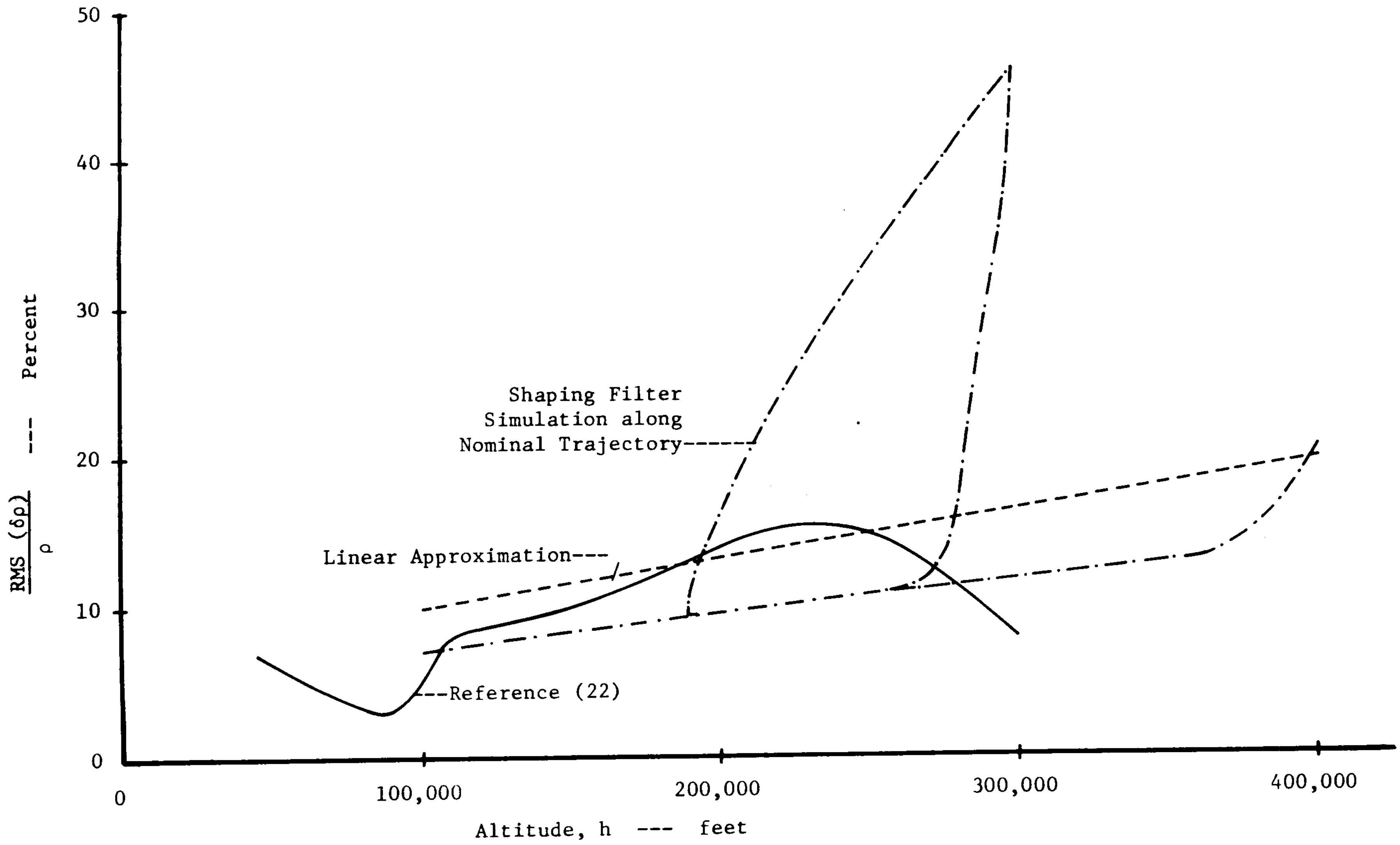


Figure 3.4 RMS Variation in Atmospheric Density

Expressing these variations as a percentage of the nominal values, f_V and f_Y, we have

$$\frac{\delta f}{f} = \frac{\delta \rho}{\rho}$$

These variations are shown in Figure 3.5 as a function of time along the nominal trajectory.

3.3.2 Random Wind Velocities

A glance at Reference (22) will reveal that the structure of atmospheric winds is indeed complex. The constantly changing pressure and temperature patterns throughout the atmosphere cause variations between two observations of wind velocities which increase with the intervals of both space and time between the observations. The rate of increase of wind changes between observations will, in turn, depend upon season, latitude, longitude, and altitude. Although very scant data is available concerning wind variations above 100,000 feet, some evidence has been shown of large day-to-day variations,of tidal variations within the period of a day, of eddies with 100 minute life span, and of small-scale turbulence with 10 to 30 second life span. Until more quantitative data is made available through analysis of rocket flights in the upper atmosphere, it will be impossible to create an accurate model of the wind variations. It would be safe to assume, however, that random variations in winds are present and could be modeled as some form of colored noise correlated with both time and distance.

For the purpose of comparison, we will investigate the effects of both down-range and cross-winds on the aerodynamic forces along the nominal trajectory chosen for study.

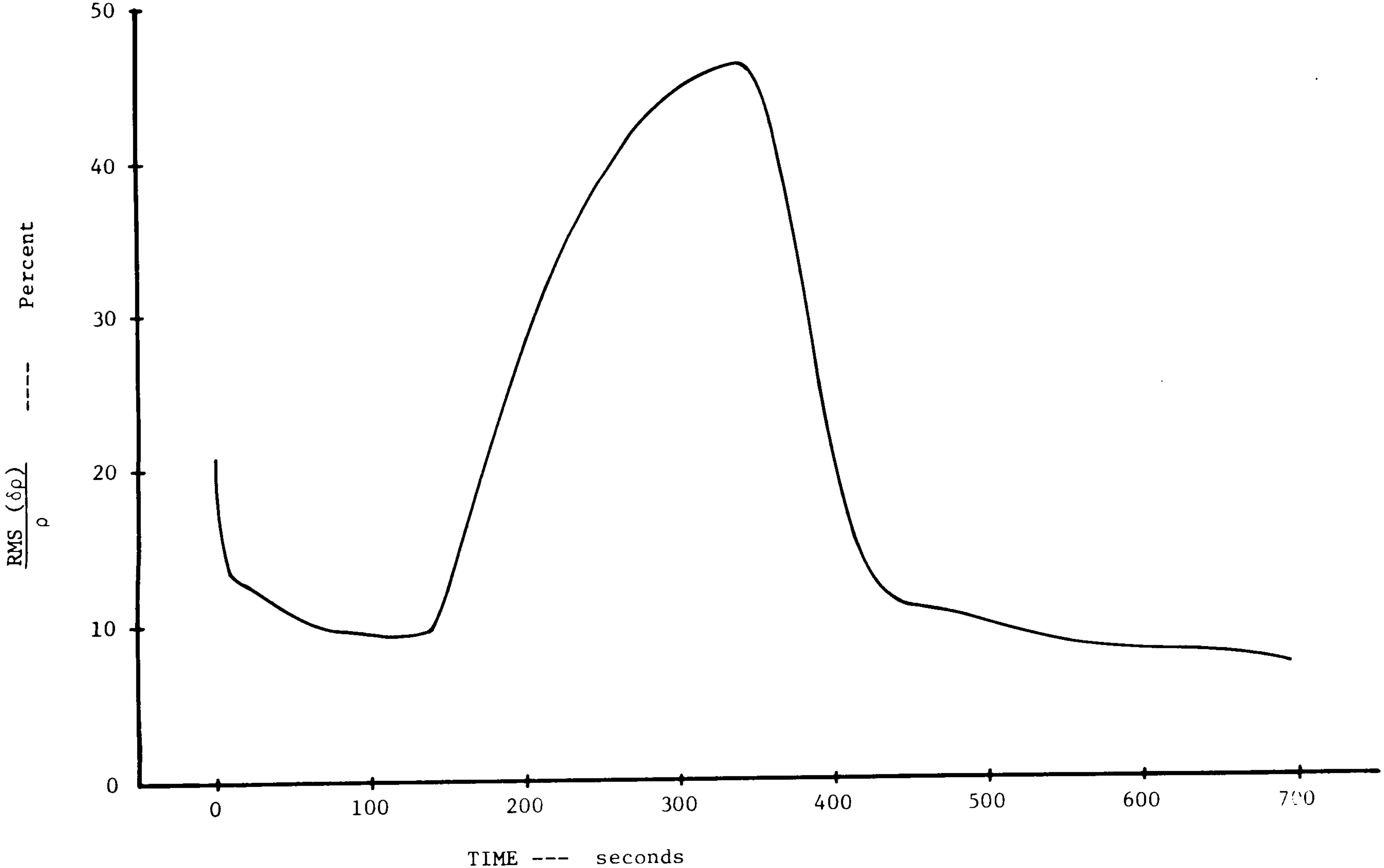


Figure 3.5 Simulation of Atmospheric Density Variations

Down range winds, w_d, may be considered as a perturbation in the velocity, v, which affects the aerodynamic forces as

$$\delta f_v = \frac{2f_v}{v} w_d$$

$$\delta f_\gamma = \frac{2f_\gamma}{v} w_d$$

Assuming that the cross winds, w_c, act in the direction of positive lift in the plane of motion, their effect on the aerodynamic forces may be written as

$$\delta f_v = -\frac{f_\gamma}{v} w_c$$

$$\delta f_\gamma = \frac{f_v}{v} w_c$$

The effect of constant wind components of 10 fps acting along the trajectory on the aerodynamic force accelerations are shown in Figure 3.6 as a percentage of the aerodynamic accelerations. We note the greatest effect of winds to be the perturbation of f_γ due to the cross winds. This effect is notable as the in-plane lift component, f_γ, approaches zero.

3.3.3 Vehicle Oscillations

As discussed in section 3.1, an on-off type control system will be employed on the Apollo command module to maintain the aerodynamically stable attitude during re-entry. Due to random aerodynamic torques about this stable attitude and to dead zones in the operation of the stabilization system, some random oscillations will be experienced by the vehicle in the pitch and yaw directions. Approximate data

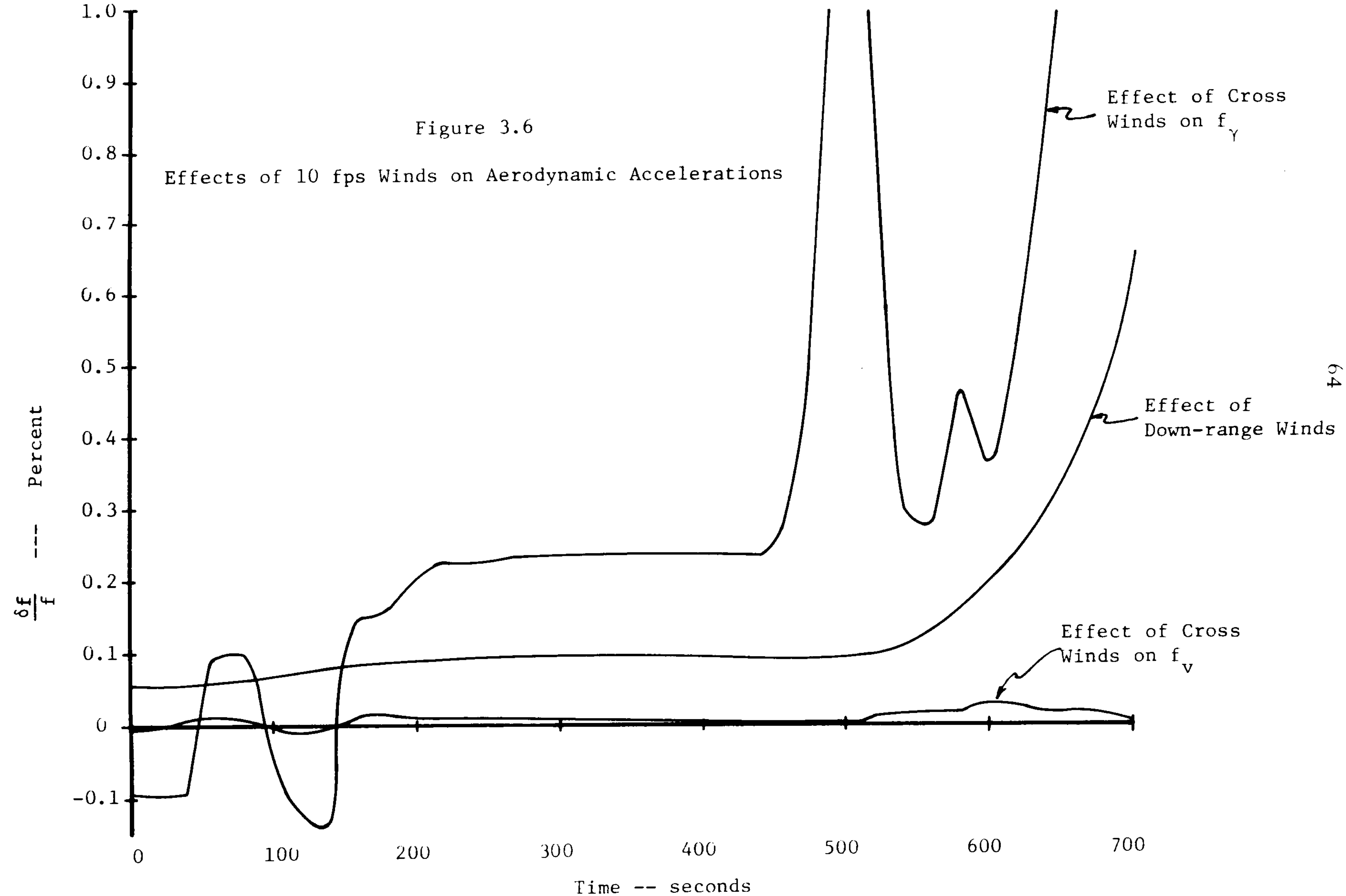


Figure 3.6
Effects of 10 fps Winds on Aerodynamic Accelerations
Effect of Cross Winds on f_γ
Effect of Down-range Winds
Effect of Cross Winds on f_v
$\frac{\delta f}{f}$ --- Percent
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
0
100
200
300
400
500
600
700
Time -- seconds

obtained from R. Morth[24] show these oscillations to have an average frequency of one cycle per second with an amplitude of one degree. In order to model these oscillations, we will treat them as white noise which, when passed through a one cycle per second band filter, produces an equivalent RMS value of

$$\text{RMS}\ (\delta\alpha) = \left[\int_0^1 (\sin (2\pi t))^2\, dt \right]^{1/2} = \frac{\sqrt{2}}{2}\ \text{degree}$$

The variation in angle of attack α and sideslip angle ζ would then be written as

$$\delta\alpha(t) = u_\alpha(t)$$

$$\delta\zeta(t) = u_\zeta(t)$$

where

$$\mathcal{E}\,[u_\alpha(t)] = 0, \quad \mathcal{E}\,[u_\zeta(t)] = 0$$

and

$$\mathcal{E}\,[u_\alpha(t)\, u_\alpha(\tau)] = q_\alpha(t)\ \delta(t-\tau)$$

$$\mathcal{E}\,[u_\zeta(t)\, u_\zeta(\tau)] = q_\zeta(t)\ \delta(t-\tau)$$

With u_α and u_ζ treated as white noise with correlation times of 1 second and RMS values of $\frac{\sqrt{2}}{2}$ degree, we obtain

$$q_\alpha = q_\zeta = 2(1) \left(\frac{\sqrt{2}}{2}\right)^2 = 1\ \text{deg.}^2\ \text{sec.}$$

We assume the pitch and yaw oscillations to be independent so that

$$\mathcal{E}\,[\,u_\alpha(t)\;\;u_\zeta(\tau)\,] = 0$$

These noise sources would have a linear effect on variations in the aerodynamic coefficients which could be obtained from (3.3) as

$$\delta C_D = 2\,C_{D1}\,\alpha u_\alpha + C_{D2}\zeta u_\zeta$$

$$\delta C_L = (C_{LO} + 3C_{L1}\alpha^2)\;u_\alpha$$

$$\delta C_Y = (C_{YO} + 3C_{Y1}\zeta^2)\;u_\zeta$$

Through linearization of (3.2) and using the nominal values of $\zeta = 0$, $\alpha = 22^{\circ}$, and $C_Y = 0$, we obtain

$$(\delta f_v)_\alpha = -\frac{2C_{D1}\alpha f_v}{C_D}\cdot u_\alpha$$

$$(\delta f_Y)_\alpha = \frac{(C_{LO} + 3C_{L1}\alpha^2)\,f_Y}{C_L}\;u_\alpha$$

$$(\delta f_v)_\zeta = 0$$

$$(\delta f_Y)_\zeta = -\frac{(C_D + C_{Y0})}{C_L}\;\tan\phi\;f_Y u_\zeta$$

For the particular vehicle described in Chapter IV, the effects of these variations on f_v and f_Y are defined as

$$\frac{(\delta f_V)_\alpha}{f_V} = .9007\ u_\alpha$$

$$\frac{(\delta f_Y)_\alpha}{f_Y} = 1.7036\ u_\alpha$$

$$\frac{(\delta f_Y)_\zeta}{f_Y} = -6.388 \tan \phi\ u_\zeta$$

These effects are shown in Figure 3.7 along the nominal trajectory. We note the effect of yaw oscillations to be similar to the effect of cross winds in the plane of motion shown in Figure 3.6.

3.3.4 Control Implementation Errors

Since no aerodynamic moments are expected about the vehicle roll axis, any variation in the roll angle would be derived from random errors in the control system. In such an on-off type control system, the major errors are introduced through limit cycling within the roll control dead zone. We will assume well designed guidance and control systems such that the magnitude of these errors produce a negligible effect upon the aerodynamic forces.

It should be noted here that, in an actual system, the vehicle attitude, e.g., the angles α, ζ, and ϕ, would be estimated by the navigation system from IMU angular measurements. Since we have assumed perfect measurements and have not considered estimation of the vehicle attitude, we must assume that the navigation system has perfect knowledge of these angles except for high frequency variations.

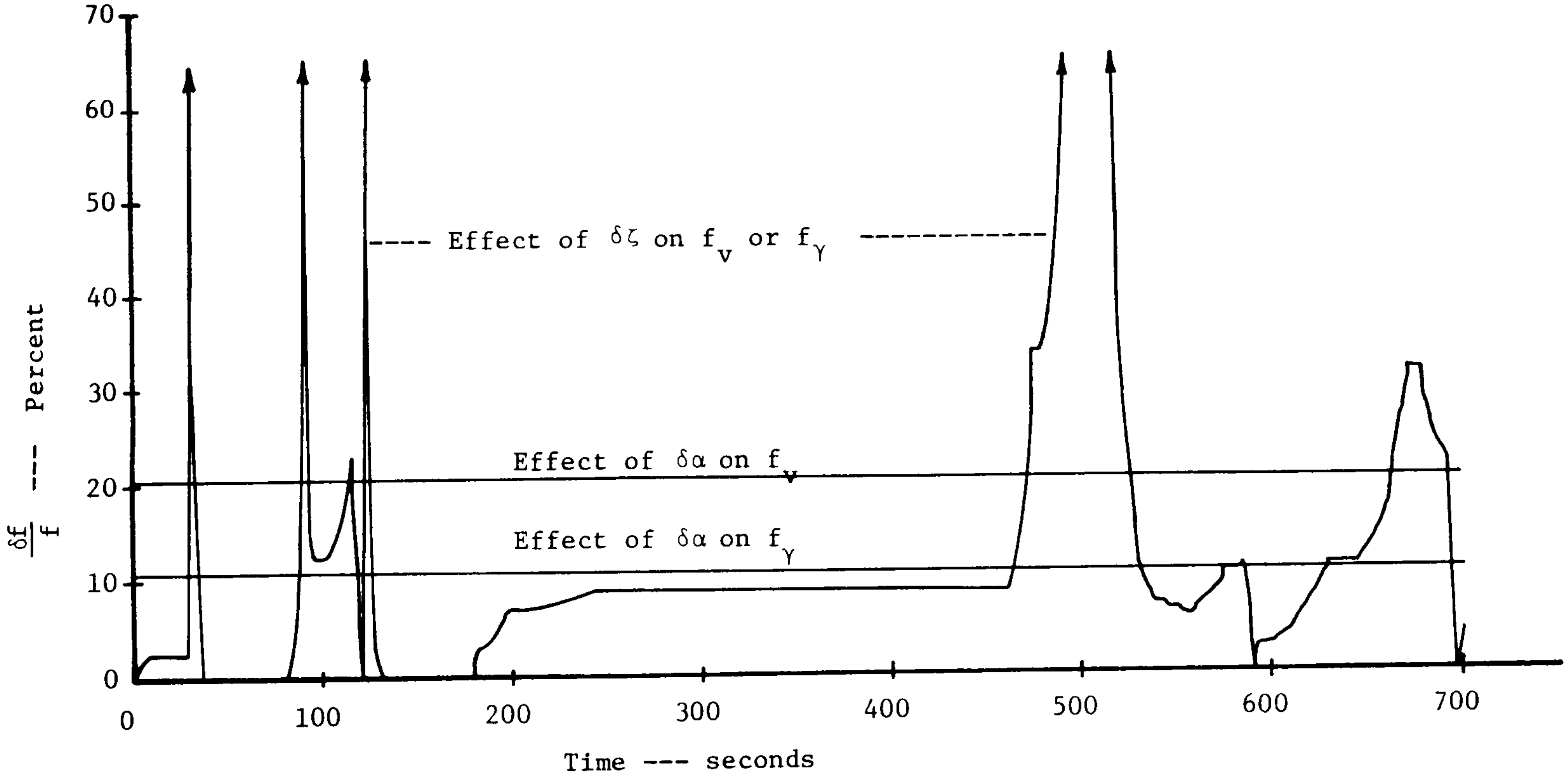


Figure 3.7 -- Variations in Aerodynamic Accelerations due to Vehicle Oscillations

3.3.5 Effects of Unsteady Flow and Mass Ablation

Among the random disturbances considered to affect the aerodynamic forces, probably the least is known about variations due to unsteady flow and mass ablation. Recent studies[24] [25] [26] have shown a marked decrease in the nominal aerodynamic coefficients due to viscous effects from asymmetric conical bodies and to local perturbations of the boundary layer and cross flow from the injection of ablative mass into the flow. The ablation was also shown in (26) to significantly change the aerodynamic pitching moment and damping coefficients. However, insufficient data is available at the present time to estimate any statistical characteristics of random variations in the aerodynamic forces or coefficients due to these effects.

Although it would seem reasonable to assume some additive white noise components in the aerodynamic coefficient variations resulting from these effects, such assumption should be made with extreme caution until some substantiating quantitative data is made available.

3.3.6 Summary

From the considerations above, it would be possible to assume that the primary sources of random disturbances in the aerodynamic forces stem from atmospheric density variations and vehicle oscillations in pitch and yaw. Although the random wind velocities, control implementation errors and mass ablation would play some role in creating random disturbances, such effects will be considered of secondary nature at the present time.

Of the three independent white noise sources considered at the present time, two enter directly into the specific force accelerations.

The third, u_ρ, also affects the aerodynamic force accelerations, but indirectly through the shaping filter for the assumed colored noise variation in atmospheric density. We define a noise vector $\underline{u}_f$ to include the two noise elements directly affecting the specific force accelerations as

$$\underline{u}_f = \begin{Bmatrix} u_\alpha \\ u_\zeta \end{Bmatrix} \tag{3.14}$$

From the statistical considerations of the independent elements in $\underline{u}_f$, we may write

$$\mathcal{E}\,[\,\underline{u}_f\,] = \underline{0}$$

and

$$\mathcal{E}\,[\,\underline{u}_f(t)\,\underline{u}_f^T(\tau)\,] = Q_f(t)\,\delta(t-\tau) \tag{3.15}$$

where

$$Q_f = \begin{bmatrix} q_\alpha & 0 \\ 0 & q_\zeta \end{bmatrix}$$

3.4 Dynamics of State Variations

From consideration of the individual sources of random variations in the state variables, it is possible to construct linear dynamic equations of motion for the total state variations, δV, $\delta\gamma$, δh, and $\delta\theta$, defined as

$$\begin{aligned} \delta V &= V - V_o \\ \delta\gamma &= \gamma - \gamma_o \\ \delta h &= h - h_o \\ \delta\theta &= \theta - \theta_o \end{aligned}$$

where the subscript o refers to the nominal values obtained from the solution of equations (3.6). Since the variation in atmospheric density, $\delta\rho$, is determined through the differential equation (3.12), we will consider the density ρ as an additional state variable in our system of equations with its differential equation written as

$$\dot{\rho} = -\beta \rho \dot{h} = -\beta \rho V \sin \gamma$$

Combining the five state variables to be considered into a vector, $\underline{s}$, defined as

$$\underline{s} = \begin{Bmatrix} V \\ \gamma \\ h \\ \theta \\ \rho \end{Bmatrix}$$

the equations describing the nominal path of the vehicle may now be written as

$$\dot{\underline{s}}_o = \begin{Bmatrix} \underline{a}_o \\ \hline \underline{0} \end{Bmatrix} + \underline{b}_o \tag{3.16}$$

where $\underline{a}_o$ is the specific force acceleration vector:

$$\underline{a}_o = \begin{Bmatrix} a_V \\ \hline a_\gamma \end{Bmatrix} = \begin{Bmatrix} f_V \\ \hline f_\gamma / V \end{Bmatrix}$$

with f_V and f_γ defined by equation (3.2), and where $\underline{b}_o$ includes the remaining non-specific force terms as

$$\underline{b}_o = \begin{Bmatrix} b_V \\ b_\gamma \\ b_h \\ b_\theta \\ b_\rho \end{Bmatrix}$$

with

$$b_V = -\frac{g_o R_e^2 \sin\gamma}{(R_e + h)^2} + (R_e + h)\,\Omega^2 \sin\gamma$$

$$b_\gamma = \frac{V\cos\gamma}{(R_e + h)} - \frac{g_o R_e^2 \cos\gamma}{V(R_e + h)^2} + 2\Omega + \frac{(R_e + h)\,\Omega^2 \cos\gamma}{V}$$

$$b_h = V\sin\gamma \tag{3.17}$$

$$b_\theta = \frac{V\cos\gamma}{(R_e + h)}$$

$$b_\rho = -\beta\rho V\sin\gamma$$

The true state vector $\underline{s}$ is found as the sum of the nominal state and random variations from the nominal, $\delta\,\underline{s}$,

$$\underline{s}(t) = \underline{s}_o(t) + \delta\,\underline{s}(t)$$

where $\delta\,\underline{s}$ will, in turn, be derived from variations in the functions $\underline{a}$ and $\underline{b}$ as

$$\delta\,\underline{s} = \begin{Bmatrix} \delta\,\underline{a} \\ \hline \underline{0} \end{Bmatrix} + \delta\,\underline{b}$$

Perturbations in the specific force accelerations will be a function of variations in the state variables and of the white noise sources considered in section 3.3. From linearization of (3.16) we obtain

$$\delta a_V = \delta f_V = a_V\left(\frac{2\delta V}{V} - \beta\delta h + \frac{\delta\rho}{\rho}\right) + \frac{\rho A_c V^2}{2m}\Big[(C_Y \cos\zeta + C_D \sin\zeta)\, u_\zeta + \sin\zeta\, \delta C_Y - \cos\zeta\, \delta C_D\Big]$$

$$\delta a_\gamma = \delta\frac{f_\gamma}{V} = a_\gamma\left(\frac{\delta V}{V} - \beta\delta h + \frac{\delta\rho}{\rho}\right) + \frac{\rho A_c V}{2m}\Big[\cos\phi\,\delta C_L - \sin\phi \sin\zeta\, \delta C_D - \sin\phi\cos\zeta\, C_Y + (C_Y \sin\phi\sin\zeta - C_D \sin\phi\cos\zeta)\, u_\zeta\Big]$$

Incorporating the variations in the aerodynamic coefficients from (3.13) and noting that the nominal value of ζ, and hence that of C_Y, is zero, we obtain

$$\delta a_V = a_V\left(\frac{2}{V}\,\delta V - \beta\delta h + \frac{1}{\rho}\,\delta\rho\right) - \frac{\rho A_c V^2}{2m}\,(2 C_{D1}\alpha u_\alpha)$$

$$\delta a_\gamma = a_\gamma\left(\frac{1}{V}\,\delta V - \beta\delta h + \frac{1}{\rho}\,\delta\rho\right) + \frac{\rho A_c V}{2m}\Big[- C_D \sin\phi\, u_\zeta + (C_{Lo} + 3\, C_{L1}\alpha^2)\, u_\alpha \cos\phi - C_{Yo}\, u_\zeta \sin\phi\Big] \tag{3.18}$$

Employing the vector notation as defined above, the total variation in the specific force acceleration may be written as

$$\delta \underline{a} = A\,\delta \underline{s} + G_f \underline{u}_f \tag{3.19}$$

where the noise vector, $\underline{u}_f$, is defined by (3.14) and the matrices A and G_f are obtained from the above linear equations as

$$A = \begin{bmatrix} \frac{2a_V}{V} & 0 & -\beta a_V & 0 & \frac{a_V}{\rho} \\ \frac{a_\gamma}{V} & 0 & -\beta a_\gamma & 0 & \frac{a_\gamma}{\rho} \end{bmatrix} \tag{3.20}$$

$$G_f = \frac{\rho A_c V}{2m} \begin{bmatrix} 2\,V\,C_{D1}\alpha & 0 \\ (C_{Lo} + 3\,C_{L1}\alpha^2)\cos\phi - (C_D + C_{Yo})\sin\phi \end{bmatrix} \tag{3.21}$$

In the same manner, we obtain variations in the non-specific force terms, $\delta\underline{b}$, as

$$\delta\underline{b} = B\,\delta\underline{s} + \underline{u}_b \tag{3.22}$$

where

$$\underline{u}_b = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_\rho \end{Bmatrix}$$

and the matrix B is obtained from linearization of equations (3.17) as

$$B = \begin{bmatrix} 0 & b_{12} & b_{13} & 0 & 0 \\ b_{21} & b_{22} & b_{23} & 0 & 0 \\ b_{31} & b_{32} & 0 & 0 & 0 \\ b_{41} & b_{42} & b_{43} & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{55} \end{bmatrix} \tag{3.23}$$

where

$$b_{12} = \left[- \frac{g_o R_e^{\;2}}{(R_e + h)^2} + (R_e + h)\,\Omega^2 \right] \cos\gamma$$

$$b_{13} = \left[\frac{2 g_o R_e^{\;2}}{(R_e + h)^3} + \Omega^2 \right] \sin\gamma$$

$$b_{21} = \left[\frac{1}{(R_e + h)} + \frac{g_o R_e^{\;2}}{V^2 (R_e + h)^2} - \frac{(R_e + h)\,\Omega^2}{V^2} \right] \cos\gamma$$

$$b_{22} = \left[- \frac{V}{(R_e + h)} + \frac{g_o R_e^{\;2}}{V (R_e + h)^2} - \frac{(R_e + h)\,\Omega^2}{V} \right] \sin\gamma$$

$$b_{23} = \left[- \frac{V}{(R_e + h)^2} + \frac{2 g_o R_e^{\;2}}{V (R_e + h)^3} + \frac{\Omega^2}{V} \right] \cos\gamma$$

$$b_{31} = \sin\gamma$$

$$b_{32} = V \cos\gamma$$

$$b_{41} = \frac{\cos\gamma}{(R_e + h)}$$

$$b_{42} = - \frac{V \sin\gamma}{(R_e + h)}$$

$$b_{43} = - \frac{V \cos\gamma}{(R_e + h)^2}$$

$$b_{55} = - \beta\, |\dot{h}| \qquad \text{(from equation (3.10))}$$

Upon combining (3.19) and (3.22), we obtain the linear differential equation for the state variations $\delta \underline{s}$ as

$$\delta \dot{\underline{s}} = F \, \delta \underline{s} + G \, \underline{u} \tag{3.24}$$

where

$$F = \begin{bmatrix} A \\ \hline O \end{bmatrix} + B$$

$$\underline{u} = \begin{Bmatrix} \underline{u}_f \\ \hline u_\rho \end{Bmatrix}$$

$$G = \left[\begin{array}{c|c} G_f & \begin{array}{c} 0 \\ 0 \end{array} \\ \hline O & \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \end{array} \right]$$

and where O is a 3x2 null matrix.

3.5 Inertial Measurement System

The proposed measurement system for the Apollo re-entry vehicle is an inertial measurement unit consisting of three single degree of freedom gyros and three accelerometers. The orientation of the accelerometer input axes, X_a, Y_a, Z_a, with respect to an inertial reference frame, X_I, Y_I, Z_I, maintained by the stable platform is shown in Figure 3.8. The inertial reference frame is established at the state of re-entry (arbitrarily set at an altitude of 400,000 feet) with the X_I axis directed radially away from the earth, the Z_I axis in the plane of motion of the vehicle normal to the X_I axis, and the Y_I axis completing the right hand triad.

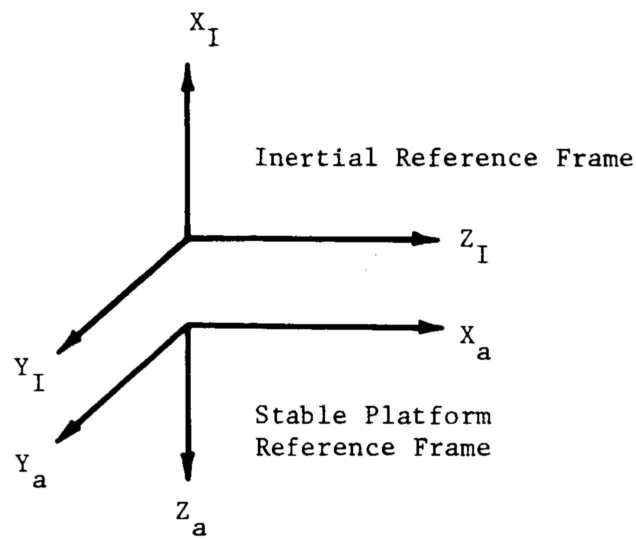


Figure 3.8

Stable Platform Orientation

Since planar motion of the vehicle is considered in the present analysis, there will be accelerations only in the X_I and Z_I directions as sensed by the (-) Z_a and X_a accelerometers, respectively. We will thus assume measurements to be received from these two accelerometers only.

The acceleration information, $\underline{a}_m$, received from the IMU in these inertial coordinates may be converted to the specific force accelerations in the rotating earth-centered coordinates employed in our model by the transformation (see Figure 3.3)

$$\underline{a}_m = T\,\underline{a} \tag{3.25}$$

where

$$T = \begin{bmatrix} \cos\chi & V\sin\chi \\ -\sin\chi & V\cos\chi \end{bmatrix}$$

and $\chi = \theta + \Omega t - \gamma$.

we note that a_γ does not represent true specific force acceleration but was defined in (3.16) as f_γ/V; hence V appears in T to transform to measured acceleration.

With the assumption that the measured accelerations $\underline{a}_m$ contain no errors from the IMU, any variations in $\underline{a}_m$ would be the direct result of perturbations in the state variables and in the specific force accelerations and could be written as

$$\underline{a}_m = \underline{a}_{mo} + \delta\underline{a}_m = T(\underline{s}_o)\,\underline{a}_o + \delta\underline{a}_m$$

where

$$\delta\underline{a}_m = \delta T\,\underline{a}_o + T(\underline{s}_o)\,\delta\underline{a} \tag{3.26}$$

Assuming that δT is the result of small perturbations in the state variables we obtain

$$\delta T = \begin{bmatrix} -\sin\chi & V\cos\chi \\ -\cos\chi & -V\sin\chi \end{bmatrix} (\delta\theta - \delta\gamma) + \begin{bmatrix} 0 & \sin\chi \\ 0 & \cos\chi \end{bmatrix} \delta V$$

Premultiplication of each term by $T\,T^{-1} = I$, where

$$T^{-1} = \begin{bmatrix} \cos\chi & -\sin\chi \\ \frac{\sin\chi}{V} & \frac{\cos\chi}{V} \end{bmatrix} \qquad (3.26a)$$

yields

$$\delta T = T \begin{bmatrix} 0 & V \\ -\frac{1}{V} & 0 \end{bmatrix} (\delta\theta - \delta\gamma) + T \begin{bmatrix} 0 & 0 \\ 0 & 1/V \end{bmatrix} \delta V$$

Subsequent post-multiplication by $\underline{a}_o$ then gives

$$\delta T \underline{a}_o = T \left[\begin{Bmatrix} V a_\gamma \\ -\frac{a_v}{V} \end{Bmatrix} (\delta\theta - \delta\gamma) + \begin{Bmatrix} 0 \\ \frac{a_\gamma}{V} \end{Bmatrix} \right] \delta V$$

This result can now be written as a linear combination of the state variations $\delta \underline{s}$, as

$$\delta T \underline{a}_o = T\, Z\, \delta \underline{s} \qquad (3.27)$$

where

$$Z = \begin{bmatrix} 0 & -V a_\gamma & 0 & V a_\gamma & 0 \\ \frac{a_\gamma}{V} & \frac{a_v}{V} & 0 & -\frac{a_v}{V} & 0 \end{bmatrix}$$

With the result of the variation in specific force acceleration, $\delta \underline{a}$ given by equation (3.19), the total variation in $\underline{a}_m$ may now be written as

$$\delta \underline{a}_m = T(Z + A)\,\delta \underline{s} + T\,G_f \underline{u}_f$$

In order to remove the nominal effects of the coordinate rotation, we will transform these variations to the navigating coordinate frame and define the measurements to be considered by the navigation system as

$$\underline{y} = T^{-1}\,\delta \underline{a}_m = (Z + A)\,\delta \underline{s} + G_f \underline{u}_f \tag{3.28}$$

3.6 Derivation of Statistical Filter

Having defined the equations describing the random linear variations in the state of the vehicle and in the inertial measurements, it is now possible to employ the results of section 2.3.2 in developing a statistical filter for obtaining a best estimate of the state variables.

The linear set of equations describing the random state variations is obtained from (3.24) as

$$\delta \underline{\dot{s}} = F\,\delta \underline{s} + G\,\underline{u}$$

Linear variations in the measurements are shown in equation (3.28) as

$$\underline{y} = (Z + A)\,\delta \underline{s} + G_f \underline{u}_f$$

where the matrices Z, A, and G_f are defined by equations (3.27), (3.20), and (3.21), respectively.

As pointed out in Chapter II, a major requirement to the use of statistical estimation theory is that the measurement variations contain independent white noise elements. We can investigate this independence through evaluation of the matrix product $G_f Q_f G_f^T$

$$G_f Q_f G_f^T = \left(\frac{\rho A_c V}{2m}\right)^2 \left[\begin{array}{c:c} (c_1 V)^2 q_\alpha & c_1 c_2 V \cos\phi\, q_\alpha \\ \hdashline c_1 c_2 V \cos\phi\, q_\alpha & (c_2 \cos\phi)^2 q_\alpha + (c_3 \sin\phi)^2 q_\zeta \end{array}\right] \tag{3.29}$$

where the constants, c_1, c_2, and c_3 are defined as

$$c_1 = 2\alpha C_{D1}$$

$$c_2 = C_{Lo} + 3C_{L1}\alpha^2$$

$$c_3 = C_D + C_{Yo}$$

The determinant of $G_f Q_f G_f^T$ can be written as

$$\left|G_f Q_f G_f^T\right| = \left(\frac{\rho A_c V}{2m}\right)^4 V^2 c_1^2 c_3^2 \sin^2\phi\, q_\alpha q_\zeta$$

If the vehicle is moving with a finite velocity through the atmosphere, this result reveals a singularity only when the sine of the roll angle ϕ passes through zero. Hence the measurements will contain independent white noise elements except when this condition is reached. During the time that $\sin\phi$ is equal to zero some linear combination of the measurements could be found which contains no noise and hence produces a perfect estimate of some combination of the state variations. Since the roll angle is allowed to assume any value set by the control system, it would be necessary for the navigation system to contain two separate filters to accommodate these two situations. We now derive these two filters. For convenience we will refer to filter A and filter B for use when $\sin\phi \neq 0$ and when $\sin\phi = 0$, respectively.

3.6.1 Derivation of Filter A ($\sin\phi \neq 0$)

When the attitude of the vehicle is such that $\sin\phi$ is not zero, the measurements $\underline{y}$ will contain independent white noise elements, and we can define (in the notation of section 2.3.2)

$$\underline{z} = H\,\delta\underline{s} + D\,\underline{u}$$

where

$$H = Z + A \tag{3.30}$$

$$D = [\,G_f\ \underline{0}\,]$$

and where $\underline{u}$ is the total noise vector defined in (3.24).

From the development in Chapter II, it is found that the best estimate of the variations, $\delta\underline{s}$, may be determined from the filter (2.72) as

$$\delta\dot{\underline{\tilde{s}}} = \mathbf{B}\,\delta\underline{\tilde{s}} + \begin{bmatrix} \underline{z} \\ \hline \underline{0} \end{bmatrix} + K\,(\underline{z} - H\,\delta\underline{\tilde{s}}) \tag{3.31}$$

where

$$\mathbf{B} = F - \begin{bmatrix} H \\ O \end{bmatrix} = \begin{bmatrix} A \\ O \end{bmatrix} + B - \begin{bmatrix} A + Z \\ O \end{bmatrix} = B - \begin{bmatrix} Z \\ O \end{bmatrix} \tag{3.32}$$

$$K = P\,H^T R^{-1}$$

$$R = G_f Q_f G_f^{\,T}$$

and where the estimation error covariance matrix is found as the solution to

$$\dot{P} = \mathbf{B}\,P + P\,\mathbf{B}^T + \begin{bmatrix} O & \underline{0} \\ \underline{0}^T & q_p \end{bmatrix} - K\,R\,K^T \tag{3.33}$$

with initial conditions

$$\delta\underline{\tilde{s}}(t_o) = \underline{0}$$

and

$$P(t_o) = \mathcal{E}\,[\,\delta\underline{s}(t_o)\,\delta\underline{s}^T(t_o)\,] \qquad (3.34)$$

It is now possible to formulate a navigation system with the use of this estimate of the variations as

$$\underline{\tilde{s}}(t) = \underline{s}_o(t) + \delta\underline{\tilde{s}}(t) \qquad (3.35)$$

where $\underline{s}_o$ is the nominal value of the state obtained from equations (3.16) and where $\delta\underline{\tilde{s}}$ is the optimal estimate of the state obtained from equation (3.31). This navigation system is shown in diagram form in Figure 3.9 on page 86.

Inputs to this system consist of the acceleration, $\underline{a}_m$, received from the IMU and the value of the roll angle, ϕ. (All of this information has been assumed perfect within this analysis.) Two simultaneous integrations are performed for the nominal value, $\underline{s}_o$, and the linear best estimate, $\delta\underline{\tilde{s}}$. The best estimate, $\underline{\tilde{s}}$, is then obtained as the sum of these values.

It is often possible to simplify such an estimation system by considering a continuous update of the nominal state $\underline{s}_o$ to conform with $\underline{\tilde{s}}$. Such an update procedure would imply that $\delta\underline{\tilde{s}}$ remains zero and would thus allow integration of a single set of equations for $\underline{\tilde{s}}$. To investigate such a possibility, we combine equations (3.31) and (3.35) to obtain

$$\dot{\tilde{\underline{s}}} = \begin{bmatrix} \underline{a}_o(\underline{s}_o) \\ \underline{0} \end{bmatrix} + \underline{b}_o(\underline{s}_o) + \mathbf{B}\,\delta\tilde{\underline{s}} + \begin{bmatrix} \underline{z} \\ \underline{0} \end{bmatrix} + K\,(\underline{z} - H\,\delta\tilde{\underline{s}})$$

From (3.26) and (3.28), we find

$$\underline{a}_m = T(\underline{s}_o)\,[\,\underline{a}_o(\underline{s}_o) + \underline{z}\,]$$

or

$$\underline{z} = T^{-1}(\underline{s}_o)\underline{a}_m - \underline{a}_o(\underline{s}_o)$$

Substitution of this into the equation above yields

$$\dot{\tilde{\underline{s}}} = \begin{bmatrix} T^{-1}(\underline{s}_o)\underline{a}_m \\ \underline{0} \end{bmatrix} + \underline{b}_o(\underline{s}_o) + \mathbf{B}\,\delta\tilde{\underline{s}} + K(T^{-1}(\underline{s}_o) - \underline{a}_o(\underline{s}_o) - H\,\delta\tilde{\underline{s}})$$

The linearity assumption allows the definition of $\underline{b}_o(\underline{s}_o)$, $\underline{a}_o(\underline{s}_o)$, and $T^{-1}(\underline{s}_o)\underline{a}_m$ as

$$\underline{b}_o(\underline{s}_o) = \underline{b}_o(\tilde{\underline{s}} - \delta\tilde{\underline{s}}) = \underline{b}_o(\tilde{\underline{s}}) - B\,\delta\tilde{\underline{s}}$$

$$\underline{a}_o(\underline{s}_o) = \underline{a}_o(\tilde{\underline{s}} - \delta\tilde{\underline{s}}) = \underline{a}_o(\tilde{\underline{s}}) - A\,\delta\tilde{\underline{s}}$$

$$T^{-1}(\underline{s}_o)\underline{a}_m = T^{-1}(\tilde{\underline{s}} - \delta\tilde{\underline{s}})\underline{a}_m = T^{-1}(\tilde{\underline{s}})\underline{a}_m + Z\,\delta\tilde{\underline{s}}$$

(where derivation of the final term, $Z\,\delta\tilde{\underline{s}}$, is similar to the derivation presented in section 3.5 for $\delta\,\underline{a}_m$).

From the definitions of $\bar{B}$ and H in equations (3.32) and (3.30), we obtain

$$\bar{B}\,\delta\underline{\tilde{s}} = B\,\delta\underline{\tilde{s}} - \begin{bmatrix} Z \\ O \end{bmatrix} \delta\underline{\tilde{s}}$$

$$H\,\delta\underline{\tilde{s}} = A\,\delta\underline{\tilde{s}} + Z\,\delta\underline{\tilde{s}}$$

With these considerations, the differential equation for $\underline{\tilde{s}}$ reduces to

$$\dot{\underline{\tilde{s}}} = \begin{bmatrix} T^{-1}(\underline{\tilde{s}})\,\underline{a}_m \\ \underline{0} \end{bmatrix} + \underline{b}_o(\underline{\tilde{s}}) + K\left(T^{-1}(\underline{\tilde{s}})\,\underline{a}_m - \underline{a}_o(\underline{\tilde{s}})\right) \tag{3.36}$$

It is thus possible to considerably simplify the original statistical navigation system to the integration of one set of state variables and the determination of the gain matrix

$$K = P\,H^T\,R^{-1}$$

through the solution of equation (3.33). The functions $\underline{a}_o(\underline{\tilde{s}})$, $\underline{b}_o(\underline{\tilde{s}})$, and $T^{-1}(\tilde{s})$ are evaluated from the non-linear equations (3.16), (3.17), and (3.26a) with the use of the best estimate, $\underline{\tilde{s}}$, of the state. The simplified navigation system is shown in Figure 3.10.

3.6.2 <u>Derivation of Filter B ($\sin\phi = 0$)</u>

When the roll angle ϕ assumes a value for which $\sin\phi = 0$, equations (3.19) and (3.21) show that the noise due to random yaw oscillations will have no affect on the linear variations in the specific force accelerations. With recognition of this fact, we may remove

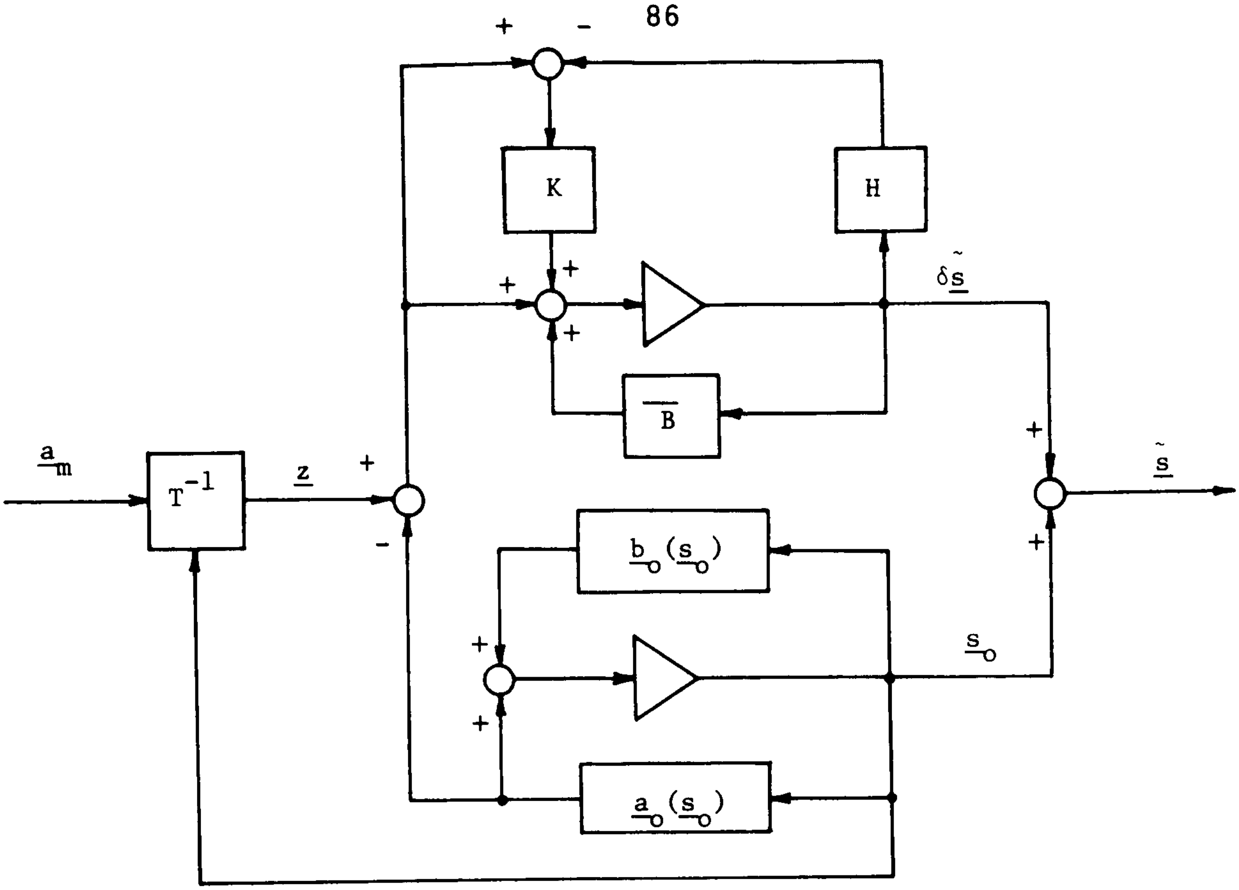


Figure 3.9 -- Diagram of Filter A

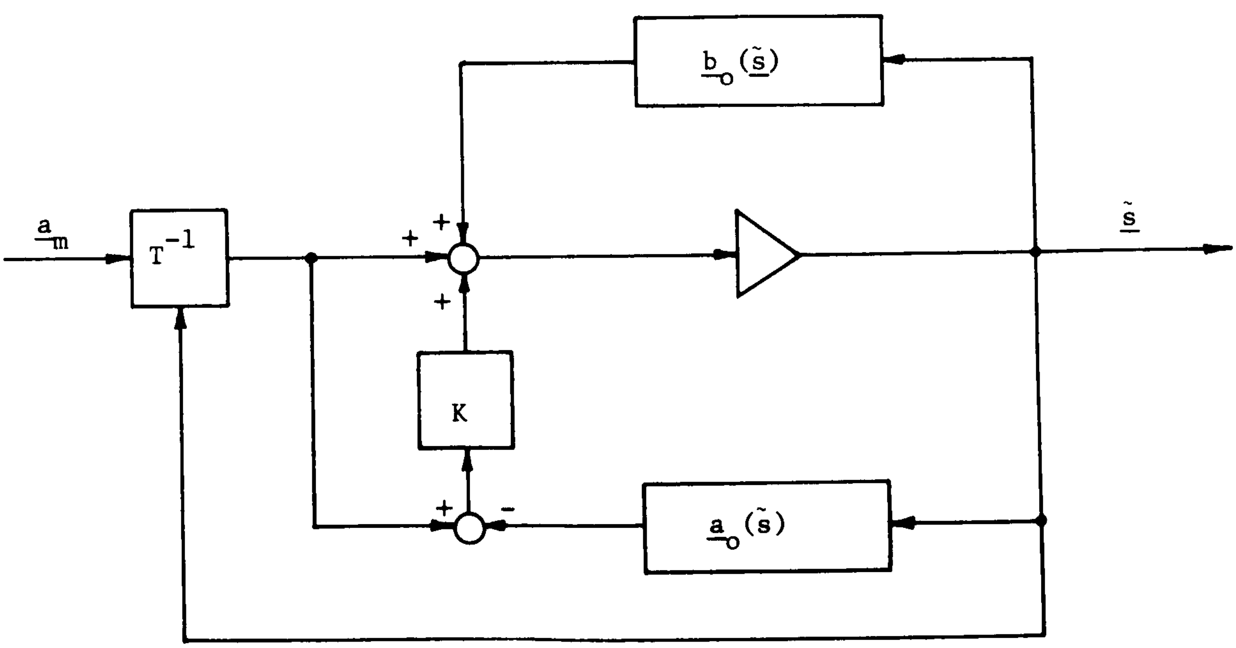


Figure 3.10 --- Simplified Diagram of Filter A

the noise element u_ζ from the system and redefine the vector $\underline{u}$ and the matrices G_f and G as

$$\underline{u} = \begin{Bmatrix} u_\alpha \\ u_\rho \end{Bmatrix} \tag{3.37}$$

$$G_f = \frac{\rho A_c V}{2m} \begin{bmatrix} 2 V C_{D1} \alpha \\ \hdashline (C_{Lo} + 3C_{L1}\alpha^2) \cos \phi \end{bmatrix}.$$

$$G = \begin{bmatrix} [\,G_f\,] & 0 \\ & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

With the definitions of the specific force accelerations

$$a_v = \frac{\rho A_c V^2 C_D}{2m} , \qquad a_\gamma = \frac{\rho A_c V C_L \cos\phi}{2m} ,$$

G_f may be written as

$$G_f = \begin{bmatrix} c_1 a_v \\ c_2 a_\gamma \end{bmatrix} \tag{3.38}$$

where the constants c_1 and c_2 have been redefined as

$$c_1 = \frac{2\,\alpha\, C_{D1}}{C_D} \quad \text{and} \quad c_2 = \frac{(C_{Lo} + 3C_{L1}\alpha^2)}{C_L} \tag{3.39}$$

The covariance of the noise vector $\underline{u}$ will be defined as

$$\mathcal{E}\,[\,\underline{u}(t)\,\underline{u}^T(\tau)\,] = Q(t)\,\delta(t-\tau) \tag{3.40}$$

where

$$Q = \begin{bmatrix} q_\alpha & 0 \\ 0 & q_\rho \end{bmatrix}$$

The linear equations describing the state and measurement variations may be written in terms of these quantities as

$$\delta\dot{\underline{s}} = F\,\delta\underline{s} + G\,\underline{u} \tag{3.41}$$

$$\underline{y} = (A + Z)\,\delta\underline{s} + G_f u_\alpha \tag{3.42}$$

where

$$F = \begin{bmatrix} A \\ O \end{bmatrix} + B$$

and where A, B, and Z are obtained from (3.20), (3.23), and (3.27) as

$$A = \begin{bmatrix} \dfrac{2\,a_V}{V} & 0 & -\beta a_V & 0 & \dfrac{a_V}{\rho} \\ \dfrac{a_Y}{V} & 0 & -\beta a_Y & 0 & \dfrac{a_Y}{\rho} \end{bmatrix} \tag{3.43}$$

$$B = \begin{bmatrix} 0 & b_{12} & b_{13} & 0 & 0 \\ b_{21} & b_{22} & b_{23} & 0 & 0 \\ b_{31} & b_{32} & 0 & 0 & 0 \\ b_{41} & b_{42} & b_{43} & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{55} \end{bmatrix} \tag{3.44}$$

$$Z = \begin{bmatrix} 0 & -V a_\gamma & 0 & V a_\gamma & 0 \\ \frac{a_\gamma}{V} & \frac{a_v}{V} & 0 & -\frac{a_v}{V} & 0 \end{bmatrix} \tag{3.45}$$

with the elements of B defined in (3.23).

Since the measurement variations, $\underline{y}$, are dependent in u_α, it is possible to obtain one linear combination of the measurement variations which contains white noise and one which is noise-free. We represent these in the notation of section 2.3.1 as

$$z_1 = L_1 \underline{y} \tag{3.46}$$

$$x_2 = L_2 \underline{y} \tag{3.47}$$

where

$$L = \begin{bmatrix} L_1 \\ \hdashline L_2 \end{bmatrix} = \begin{bmatrix} \ell_1 & \ell_2 \\ \ell_3 & \ell_4 \end{bmatrix} \tag{3.48}$$

is a non-singular linear transformation such that

$$L_2 G_f u_\alpha = 0 \tag{3.49}$$

We thus obtain perfect knowledge of

$$x_2 = L_2 (A + Z) \delta \underline{s} = M_2 \delta \underline{s} \tag{3.50}$$

as the linear combination M_2 of the state variations. Differentiating x_2, we obtain

$$\dot{x}_2 = \dot{M}_2 \delta \underline{s} + M_2 \delta \underline{\dot{s}}$$

or

$$\dot{x}_2 = (\dot{M}_2 + M_2 F) \delta \underline{s} + M_2 G \underline{u} \tag{3.51}$$

If $M_2 G \underline{u}$ and $L_1 G_f u_\alpha$ contain independent white noise elements, we can define a new measurement $z_2 = \dot{x}_2$ and proceed with the definition of the statistical filter corresponding to the measurements

$$\underline{z} = \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} L_1 \underline{y} \\ \dot{x}_2 \end{Bmatrix} = H_1 \, \delta \underline{s} + D \underline{u} \qquad (3.52)$$

where

$$H_1 = \left[\begin{array}{c} L_1(A + Z) \\ \hline \dot{M}_2 + M_2 F \end{array}\right] \qquad (3.53)$$

and

$$D = \left[\begin{array}{c:c} L_1 G_f & \underline{0} \\ \hline \multicolumn{2}{c}{M_2 G} \end{array}\right]$$

Independence of noise in $\underline{z}$ will exist if the determinant of DQD^T can be shown to be non-singular. To this end, we derive the components of the D matrix. From (3.48) and (3.38) we obtain

$$L_1 G_f = \ell_1 c_1 a_v + \ell_2 c_2 a_\gamma \qquad (3.54)$$

The requirement (3.49) that $L_2 G_f u_\alpha = 0$, provides a relationship between ℓ_3 and ℓ_4 such that

$$\ell_4 = - \frac{c_1 a_v}{c_2 a_\gamma} \ell_3 \qquad (3.55)$$

Also, the determinant of L can be written as

$$|L| = \ell_1 \ell_4 - \ell_2 \ell_3 = - \frac{\ell_3}{c_2 a_\gamma} (\ell_1 c_1 a_v + \ell_2 c_2 a_\gamma) \qquad (3.56)$$

Using this result in (3.54), there follows

$$L_1 G_f = - \frac{c_2 a_\gamma |L|}{\ell_3} \tag{3.57}$$

From (3.50) we obtain

$$M_2 = L_2 (A + Z)$$

or

$$M_2 = \ell_3 a_v \left[\frac{2 c_o}{V} - W - \beta c_o \quad W \frac{c_o}{\rho} \right] \tag{3.58}$$

where $$c_o = 1 - \frac{c_1}{c_2} \tag{3.59}$$

and $$W = \frac{V a_\gamma}{a_v} + \frac{c_1}{c_2} \frac{a_v}{V a_\gamma} \tag{3.60}$$

We note that when $\sin \phi = 0$, $\frac{V a_\gamma}{a_v} = \pm \frac{C_L}{C_D} = \pm .3$, so that $W =$ constant.

The D matrix may now be written in terms of the above definitions as

$$D = \left[\begin{array}{c|c} - \dfrac{c_2 a_\gamma |L|}{\ell_3} & 0 \\ \hline \ell_3 c_2 a_v a_\gamma \left(\dfrac{2 c_o c_1 a_v}{V c_2 a_\gamma} - W \right) & \dfrac{\ell_3 a_v c_o}{\rho} \end{array} \right] \tag{3.61}$$

If Q is a positive definite matrix, the product DQD^T will be nonsingular if the determinant of D is non-zero. From the above relationship, we write

$$|D| = - \frac{c_o c_2 a_v a_\gamma |L|}{\rho} = \frac{(c_1 - c_2) a_v a_\gamma |L|}{\rho}$$

Hence, if $c_1 \neq c_2$, and L is non-singular, the measurements $\underline{z}$ as defined by (3.52) will contain independent white noise. We may now continue with the derivation of the filter for these measurements according to the theory developed in section 2.3.2.

We note that perfect knowledge of a linear combination of $\delta\,\underline{s}$ is available through the relationship $x_2 = M_2 \delta\,\underline{s}$. Thus the estimation of $\delta\,\underline{s}$ will be reduced to the estimation of

$$\underline{x}_1 = M_1 \delta\,\underline{s} \tag{3.62}$$

where M_1 is chosen such that

$$M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \tag{3.63}$$

is non-singular. From knowledge of M_2 from (3.58), we may choose M_1 as

$$M_1 = [\,I \quad O\,] \tag{3.64}$$

so that

$$\underline{x}_1 = \begin{Bmatrix} \delta V \\ \delta \gamma \\ \delta h \\ \delta \theta \end{Bmatrix}$$

Noting that the matrix D is invertible, it is possible to estimate $\underline{x}_1$ by the filter (2.64) as

$$\dot{\tilde{\underline{x}}}_1 = \bar{F}\,\tilde{x}_1 + F_{12}x_2 + \bar{K}(\bar{\underline{z}} - \bar{H}\tilde{x}_1) + G_1\,\bar{\underline{z}} \tag{3.65}$$

where

$$\overline{F} = F_{11} - G_1 \overline{H}$$

$$\overline{H} = D^{-1} H_1 M^{-1} M_1^{T}$$

$$G_1 = M_1 G$$

$$[\,F_{11}\; F_{12}\,] = \dot{M} M^{-1} + M F M^{-1}$$

$$\overline{\underline{z}} = D^{-1} \underline{z} - D^{-1} H_1 M^{-1} \left\{ \frac{0}{x_2} \right\}$$

$$\overline{K} = P_1 \overline{H}^{T} Q^{-1} \tag{3.66}$$

and where

$$\dot{P}_1 = \overline{F} P_1 + P_1 \overline{F}^{T} - \overline{K} Q \overline{K}^{T} \tag{3.67}$$

After a significant amount of algebraic simplification, the above matrices may be written as

$$\overline{F} = M_1 \overline{B} M_1^{T} \tag{3.68}$$

where $\overline{B} = B - \begin{bmatrix} Z \\ O \end{bmatrix}$

$$F_{12} = \begin{bmatrix} \dfrac{1}{\ell_3 c_o} \\ \dfrac{a_\gamma}{a_v \ell_3 c_o} \\ 0 \\ 0 \end{bmatrix}$$

$$G_1 = \begin{bmatrix} c_1 a_V & 0 \\ c_2 a_\gamma & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$D^{-1} = \begin{bmatrix} -\dfrac{\ell_3}{c_2 a_\gamma |L|} & 0 \\ \dfrac{\ell_3 \rho}{c_o |L|}\left(\dfrac{2c_o}{V}\,\dfrac{c_1 a_V}{c_2 a_\gamma} - W\right) & \dfrac{\rho}{\ell_3 a_V c_o} \end{bmatrix}$$

$$H_1 M^{-1} \begin{Bmatrix} 0 \\ \hline x_2 \end{Bmatrix} = \begin{bmatrix} \dfrac{\ell_1 a_V + \ell_2 a_\gamma}{\ell_3 c_o a_V} \\ \dfrac{2\,a_V}{V} + \dfrac{2\dot{V}}{V} + \dfrac{\dot{\rho}}{\rho} - \dfrac{W a_\gamma}{c_o} \end{bmatrix} x_2$$

$$\bar{H} = \begin{bmatrix} 0 & \bar{h}_{12} & 0 & -\bar{h}_{12} \\ \bar{h}_{21} & \bar{h}_{22} & \bar{h}_{23} & \bar{h}_{24} \end{bmatrix} \tag{3.69}$$

where $\bar{h}_{12} = \dfrac{1}{c_2 - c_1}\left(\dfrac{a_V}{V\,a_\gamma} + \dfrac{V\,a_\gamma}{a_V}\right)$

$$\bar{h}_{21} = \frac{\rho}{V}\left[\frac{W}{c_o}\left((b_{41} - b_{21})\,V + a_\gamma\right) - \frac{2\dot{V}}{V} - \beta|\dot{h}|\right]$$

$$\bar{h}_{22} = -\,\bar{h}_{24} + \rho\left[\frac{W}{c_o}\,(b_{42} - b_{22}) + \frac{2b_{12}}{V} - \beta\,b_{32}\right]$$

$$\bar{h}_{23} = \rho \left[\frac{2(b_{13} + \beta a_v)}{V} + \frac{W}{c_o} (b_{43} - b_{23} - \beta a_\gamma) \right]$$

$$\bar{h}_{24} = -\rho \left[\frac{W}{c_o} \frac{a_v}{V} + 2 a_\gamma + \left(\frac{V a_\gamma}{a_v} - \frac{c_1 a_v}{c_2 a_\gamma V} \right) \frac{\tan\phi \dot{\phi}}{c_o} \right]$$

The best estimate of the state variations are thus obtained as

$$\delta \tilde{\underline{s}}(t) = M^{-1}(t) \begin{Bmatrix} \tilde{\underline{x}}_1(t) \\ \tilde{x}_2(t) \end{Bmatrix}$$

where

$$M^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{2\rho}{V} & \frac{W\rho}{c_o} & \beta\rho & -\frac{W\rho}{c_o} & \frac{\rho}{\ell_3 c_o a_v} \end{bmatrix}$$

and where $\tilde{\underline{x}}_1(t)$ is determined through solution of (3.65). A diagram representing this filter is shown in Figure 3.11 on page 99.

The formulation of this filter as shown above provides a relatively simple equation (3.67) for solution of the covariance matrix P. However, the filtering equation (3.65) would require much simplification before being applied in an on-board navigation system.

Noting that

$$x_2 = L_2 \underline{y},$$

we can obtain

$$F_{12}x_2 = F_{12}L_2\underline{y} = \frac{1}{c_o}\begin{bmatrix} 1 & -\dfrac{c_1 a_v}{c_2 a_\gamma} \\ \dfrac{a_\gamma}{a_v} & -\dfrac{c_1}{c_2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\underline{y}$$

Also, noting that

$$G_1 D^{-1} = \begin{bmatrix} -\dfrac{\ell_3 c_1 a_v}{c_2 a_\gamma |L|} & 0 \\ -\dfrac{\ell_3}{|L|} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and that $z_1 = L_1\underline{y}$,

we obtain

$$G_1\overline{z} = G_1 D^{-1}\left(\underline{z} - H_1 M^{-1}\begin{Bmatrix} 0 \\ \hline x_2 \end{Bmatrix}\right)$$

or

$$G_1\overline{z} = \begin{bmatrix} -\dfrac{\ell_3 c_1 a_v}{c_2 a_\gamma |L|} \\ -\dfrac{\ell_3}{|L|} \\ 0 \\ 0 \end{bmatrix}\left[L_1 - \frac{(\ell_1 a_v + \ell_2 a_\gamma)}{\ell_3 c_o a_v} L_2\right]\underline{y}$$

After some algebraic simplification, we obtain

$$F_{12}x_2 + G_1 \underline{\bar{z}} = \begin{Bmatrix} \underline{y} \\ \underline{0} \end{Bmatrix} = \begin{bmatrix} T^{-1}(s_o)\underline{a}_m - \underline{a}_o(s_o) \\ \underline{0} \end{bmatrix}$$

We now define a diminished state vector $\underline{s}_1$ as

$$\underline{s}_1 = \begin{Bmatrix} V \\ \gamma \\ h \\ \theta \end{Bmatrix}$$

and its associated best estimate as

$$\underline{\tilde{s}}_1 = \underline{s}_{1_o} + \delta \underline{\tilde{s}}_1$$

From our choice of M_1 in (3.64), we find that

$$\delta \underline{s}_1 = \underline{x}_1$$

so that $\delta \underline{\tilde{s}}_1$ is described by the filtering equation (3.65). Through considerations similar to those used to simplify filter A, it is possible to obtain a differential equation for $\underline{\tilde{s}}_1$ as

$$\dot{\underline{\tilde{s}}}_1 = \begin{Bmatrix} T^{-1}(\underline{\tilde{s}}_1)\,\underline{a}_m \\ \underline{0} \end{Bmatrix} + \underline{b}_1(\underline{\tilde{s}}_1) + \bar{K} D^{-1} \begin{bmatrix} L_1 \underline{\tilde{y}} \\ \frac{d}{dt}(L_2 \underline{\tilde{y}}) \end{bmatrix}$$

where

$$\underline{b}_1 = \begin{Bmatrix} b_V \\ b_\gamma \\ b_h \\ b_\theta \end{Bmatrix} \quad \text{as obtained from (3.17)}$$

and

$$\underline{\tilde{y}} = T^{-1}(\underline{\tilde{s}}_1)\underline{a}_m - \underline{a}_o(\underline{\tilde{s}}_1, \tilde{\rho})$$

with the best estimate of ρ obtained as

$$\dot{\tilde{\rho}} = \underline{b}_\rho(\underline{\tilde{s}}) + \frac{\rho}{\ell_3 a_v c_o} \frac{d}{dt}(L_2 \underline{\tilde{y}})$$

We note that the elements of the linear transformation matrix L will always be eliminated in the final results. Hence, no generality is lost by choosing $\ell_3 = 1$.

The entire filter B for $\underline{\tilde{s}}(t)$ may then be written as

$$\dot{\underline{\tilde{s}}} = \begin{Bmatrix} T^{-1}(\underline{\tilde{s}})\underline{a}_m \\ \underline{0} \end{Bmatrix} + \underline{b}_o(\underline{\tilde{s}}) + \bar{K}' \begin{bmatrix} L_1 \underline{\tilde{y}} \\ \frac{d}{dt}(L_2 \underline{\tilde{y}}) \end{bmatrix} \qquad (3.70)$$

where

$$\bar{K}' = \left[\begin{array}{c|c} \multicolumn{2}{c}{\bar{K}\,D^{-1}} \\ \hline 0 & \frac{\rho}{a_v c_o} \end{array}\right]$$

A diagram of this simplified filter is shown in Figure 3.12. We note a similarity with Filter A as described by equation (3.36) except for the differentiation of measurements.

3.6.3 Combined Dual Filtering Navigation System

A complete statistical navigation system for the random errors considered in section 3.3 and for assumed perfect measurements would require the use of filters A and B defined above and additional switching logic to provide continuous navigation as the roll angle passes through

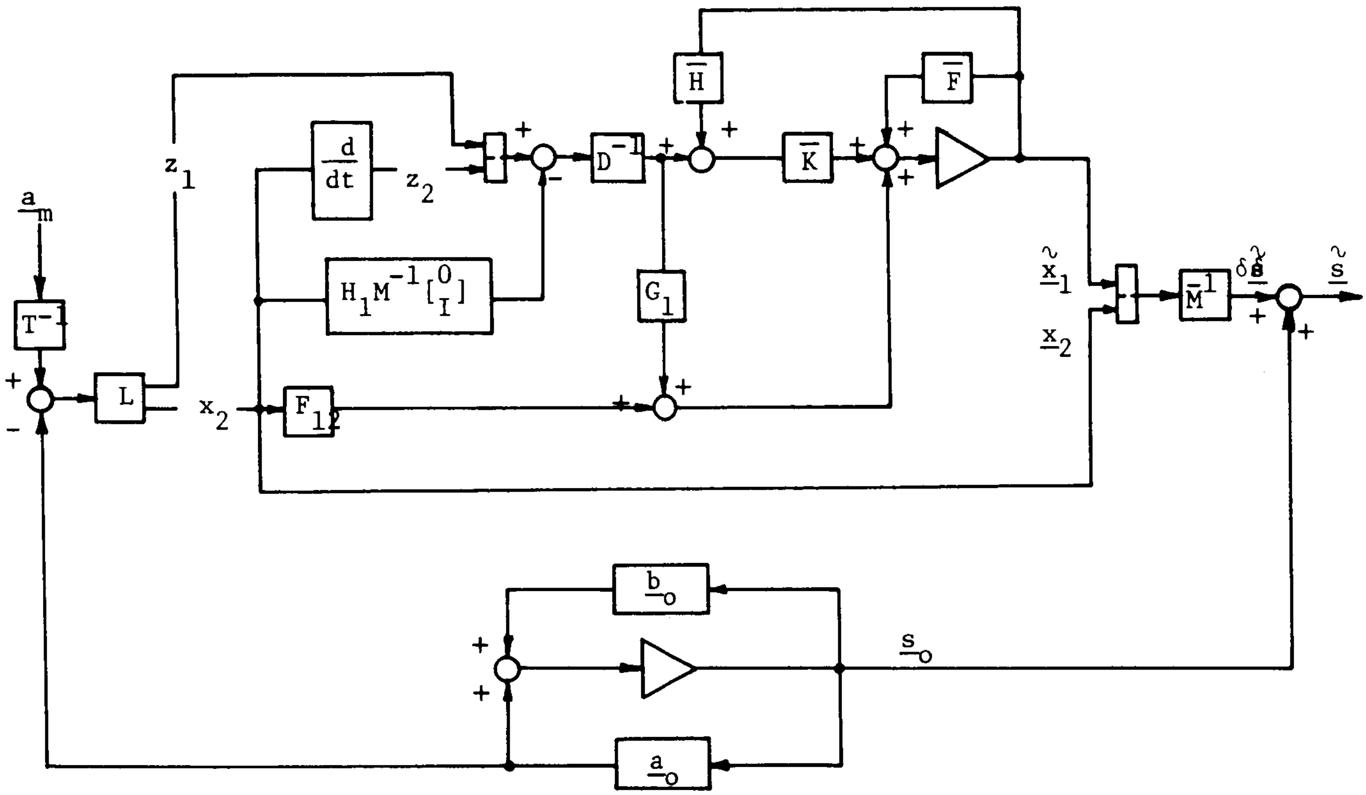


Figure 3.11

Diagram of Filter B

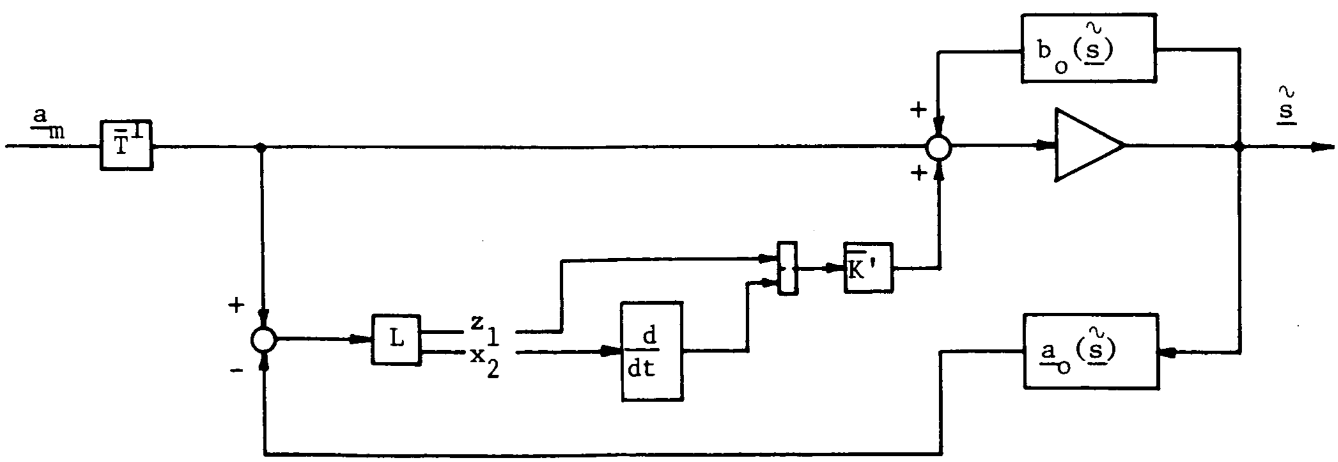


Figure 3.12

Simplified Filter B

0 or 180 degrees. The switching logic must select the times at which switching is to occur and equip each filter with appropriate initial conditions.

Upon switching from filter A to filter B at time t_o, the initial conditions for filter B are obtained from equations (2.51) and (2.52) as

$$\underline{\tilde{s}}(t_o+) = \underline{\tilde{s}}(t_o) + M^{-1} \begin{Bmatrix} \underline{x}_1(t_o+) \\ \underline{x}_2(t_o) \end{Bmatrix}$$

$$= \underline{\tilde{s}}(t_o) + M^{-1} \begin{bmatrix} M_1 P(t_o) M_2^T(t_o) \left[M_2(t_o) P(t_o) M_2^T(t_o) \right]^{-1} \\ 1 \end{bmatrix} x_2$$

$$P_1(t_o+) = M_1 P(t_o+)\, M_1^T \tag{3.71}$$

where

$$P(t_o+) = P(t_o) - P(t_o) M_2^T(t_o) \left[M_2(t_o) P(t_o) M_2^T(t_o) \right]^{-1} M_2(t_o) P(t_o) \tag{3.72}$$

and where $P(t_o)$ is the estimation error covariance matrix obtained from filter A immediately before switching.

Upon switching back to filter A at time t_1, the initial value of $P(t_1)$ is obtained from $P_1(t_1)$ according to equation (2.55) as

$$P(t_1) = M^{-1}(t_1) \begin{bmatrix} P_1(t_1) & \underline{0} \\ \underline{0}^T & 0 \end{bmatrix} M^{-1\,T}(t_1) \tag{3.73}$$

The switching times, t_o and t_1 must be chosen with discretion. Since the error covariance matrix R in equation (3.32b) for filter A becomes singular as $\sin\phi$ approaches zero, the gain matrix K employed in equation (3.33) for $\dot{P}$ will become indefinite when this condition is reached. Hence, switching should occur at times when $\sin\phi = \epsilon$, where ϵ is a small number.

The perfect measurement realized when $\sin\phi = 0$ affords a very pronounced reduction of the estimation errors. As suggested by (3.68) and (3.69), the extent of this reduction is to reduce the rank of the covariance matrix P by one. Since the sine of the roll angle must be continuous as the vehicle is rotated, however, we should expect the covariance matrix for filter A to approach this singularity condition before filter B is brought into use. In order to determine to some extent how this reduction is initiated, we examine the propagation of P from equation (3.33) for small values of $\sin\phi$.

With the assurance that the tangent of ϕ will not be infinite in this region, we can rewrite the measurement error covariance matrix R from equation (3.29) as

$$R = \begin{bmatrix} (c_1 a_v)^2 q_\alpha & c_1 c_2 a_v a_\gamma q_\alpha \\ c_1 c_2 a_v a_\gamma q_\alpha & a_\gamma^2 (c_2^2 q_\alpha + c_3^2 \tan^2\phi\, q_\zeta) \end{bmatrix}$$

where

$$c_1 = \frac{2\alpha C_{D1}}{C_D}$$

$$c_2 = \frac{C_{Lo} + 3C_{L1}\alpha^2}{C_L} \qquad c_3 = \frac{C_D + C_{Yo}}{C_L}$$

or equivalently as

$$R = \begin{bmatrix} a_v & 0 \\ 0 & a_\gamma \end{bmatrix} \begin{bmatrix} c_1^2 q_\alpha & c_1 c_2 q_\alpha \\ c_1 c_2 q_\alpha & c_2^2 q_\alpha + c_3^2 \tan^2 \phi q_\zeta \end{bmatrix} \begin{bmatrix} a_v & 0 \\ 0 & a_\gamma \end{bmatrix}$$

The inverse of R can then be written as

$$R^{-1} = \begin{bmatrix} \frac{1}{a_v} & 0 \\ 0 & \frac{1}{a_\gamma} \end{bmatrix} \begin{bmatrix} \frac{1}{c_1^2}\left(\frac{c_2^2}{c_3^2 \tan^2 \phi q_\zeta} + \frac{1}{q_\alpha}\right) & -\frac{c_2}{c_1 c_3^2 \tan^2 \phi q_\zeta} \\ -\frac{c_2}{c_1 c_3^2 \tan^2 \phi q_\zeta} & \frac{1}{c_3^2 \tan^2 \phi q_\zeta} \end{bmatrix} \begin{bmatrix} \frac{1}{a_v} & 0 \\ 0 & \frac{1}{a_\gamma} \end{bmatrix}$$

Noting that

$$H = Z + A = \begin{bmatrix} \frac{2a_v}{V} & -V a_\gamma & 0 & V a_\gamma & \frac{a_v}{\rho} \\ \frac{2a_\gamma}{V} & \frac{a_v}{V} & 0 & -\frac{a_v}{V} & \frac{a_\gamma}{\rho} \end{bmatrix}$$

the matrix product $H^T R^{-1} H$, for small values of $\tan\phi$, can be written as

$$H^T R^{-1} H = \frac{1}{q_\zeta \tan^2 \phi} \, \mathbf{E}$$

where

$$\bar{E} = \begin{bmatrix} \frac{4}{V^2} k_1 & \frac{2}{V} k_2 & 0 & -\frac{2}{V} k_2 & \frac{2k_1}{\rho V} \\ \frac{2}{V} k_2 & k_3 & 0 & -k_3 & \frac{1}{\rho} k_2 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{2}{V} k_2 & -k_3 & 0 & k_3 & -\frac{k_2}{\rho} \\ \frac{2}{\rho V} k_1 & \frac{k_2}{\rho} & 0 & -\frac{k_2}{\rho} & \frac{k_1}{\rho^2} \end{bmatrix}$$

with $k_1 = \frac{1}{c_3^2} \left(\frac{c_2}{c_1} - 1 \right)^2, \; k_2 = \frac{1}{c_3^2} \left(1 - \frac{c_2}{c_1} \right) \left(\frac{C_D}{C_L} + \frac{c_2}{c_1} \frac{C_L}{C_D} \right),$

$$k_3 = \frac{1}{c_3^2} \left(\frac{C_L^2}{C_D^2} \frac{c_2^2}{c_1^2} + 2 \frac{c_2}{c_1} + \frac{C_D^2}{C_L^2} \right)$$

The final term in (3.33) may thus be approximated as

$$K R K^T = P H^T R^{-1} H P = \frac{1}{q_\zeta \tan^2 \phi} P \bar{E} P$$

As $\tan \phi$ approaches zero, we would expect this term to be the dominating term in equation (3.33) and $\dot{P}$ could then be approximated as

$$\dot{P} \cong -\frac{1}{q_\zeta \tan^2 \phi} P \bar{E} P$$

Hence, we find that, as the condition $\sin \phi = 0$ is approached while employing filter A, the estimation errors for all the state variables will be reduced. Also, if the filtering is continued to $\sin\phi = 0$, the errors in estimation observed at this condition should theoretically be exactly zero. This trend is observed in the computer simulation discussed in Chapter IV.

Note should also be taken of the manner in which the atmospheric density noise has been defined in section 3.3.1. In order to transform the altitude dependent white noise function $w_\rho(h)$ to time dependent noise, $u_\rho(t)$, it was assumed permissible to multiply by the altitude rate as

$$u_\rho(t) = |\dot{h}|\ w_\rho(h)$$

This transformation would imply that perfect knowledge of the atmospheric density is obtained when the vehicle is travelling at constant altitude and would hence remove the effect of u_ρ when $\dot{h}$ is zero. The result of this removal would have little effect on the operation of filter A, since it would merely remove the positive forcing term q_ρ from equation (3.33). However, in filter B, the removal of u_ρ would again suggest measurements which are dependent in white noise and thus force a redefinition of the measurement and filtering equations. Hence the possibility of the two conditions $\sin\phi = 0$, and $\dot{h} = 0$, would have to be considered in the design of a practical navigation system.

3.7 Additional White Noise Considerations

The dual filtering system derived above appears to be somewhat cumbersome and impractical for employment as a real-time on-board navigation system due to the necessary switching logic and to the requirement for two independent filtering systems.

The need for a dual filter is created by the removal of the effects of primary noise elements from the measurements when such conditions as zero roll angle and zero altitude rate are realized. This ineffectiveness of the primary noise sources, however, would suggest that some additional white noise, neglected because of its secondary nature, would have a dominant effect on the system when the roll angle is zero.

The consideration of any additional white noise sources in the aerodynamic forces or in the measurement system would provide the assurance of independent white noise in the measurements at all times. Thus, the need for filter B would be eliminated and a simplified navigation system would be obtained with the employment of a single filter.

In order to compare such a single filtering system with the dual system derived above, we will consider some additive white noise in the measurements such that equation (3.28) becomes

$$\underline{y} = (Z + A)\,\delta\underline{s} + G_f\underline{u}_f + G_a\underline{u}_a \tag{3.74}$$

where $\underline{u}_a$ is white noise with zero mean and covariance

$$\mathcal{E}\,[\,\underline{u}_a(t)\,\underline{u}_a(\tau)^T\,] = Q_a(t)\,\delta(t-\tau) \tag{3.75}$$

Without physical justification for the origin of this noise either in the measurement system or in the aerodynamic forces, we will assume it to be obtained entirely from high frequency random uncertainties in the measurement data. From considerations in section 3.5, then, we obtain the matrix G_a as

$$G_a = T^{-1} = \begin{bmatrix} \cos\chi & -\sin\chi \\ \frac{\sin\chi}{V} & \frac{\cos\chi}{V} \end{bmatrix} \tag{3.76}$$

We also assume the matrix Q_a to be diagonal with equal elements as

$$Q_a = \begin{bmatrix} q_a & 0 \\ 0 & q_a \end{bmatrix}$$

Defining a vector of noise elements affecting the measurements as

$$\underline{u}_m = \begin{Bmatrix} \underline{u}_f \\ \underline{u}_a \end{Bmatrix} \quad ,$$

the measurement variations may be written as

$$\underline{y} = (Z + A)\,\delta\,\underline{s} + [\, G_f \,\vdots\, G_a \,]\ \underline{u}_m$$

If we assume no correlation between $\underline{u}_f$ and $\underline{u}_a$, the covariance matrix of $\underline{u}_m$ becomes

$$\mathcal{E}\,[\,\underline{u}_m(t)\,\underline{u}_m(\tau)^T] = \begin{bmatrix} Q_f & 0 \\ 0 & Q_a \end{bmatrix} \delta(t-\tau) = Q_m\ \delta(t-\tau)$$

and hence the matrix R_s, representing the covariance of measurement noise, is determined as

$$R_s = [\, G_f \,\vdots\, G_a \,]\ Q_m \begin{bmatrix} G_f^{\,T} \\ G_a^{\,T} \end{bmatrix} = G_f Q_f G_f^{\,T} + G_a Q_a G_a^{\,T} \tag{3.77}$$

We can now investigate the independence of white noise elements in the measurements through evaluation of the singularity of R_s. With G_a and Q_a defined above, we obtain

$$G_a Q_a G_a^T = \begin{bmatrix} q_a & 0 \\ 0 & \dfrac{q_a}{V^2} \end{bmatrix} \tag{3.78}$$

Adding this to $G_f Q_f G_f^T$ defined by (3.29), R_s becomes

$$R_s = \begin{bmatrix} \left(\dfrac{c_1 \rho A_c V^2}{2m}\right)^2 q_\alpha + q_a & \left(\dfrac{\rho A_c V}{2m}\right)^2 c_1 c_2 V \cos\phi q_\alpha \\ \left(\dfrac{\rho A_c V}{2m}\right)^2 c_1 c_2 V \cos\phi q_\alpha & \left(\dfrac{\rho A_c V}{2m}\right)^2 \left[(c_2 \cos\phi)^2 q_\alpha + (c_3 \sin\phi)^2 q_\zeta\right] + \dfrac{q_a}{V^2} \end{bmatrix} \tag{3.79}$$

The term $\dfrac{q_a^2}{V^2}$ in the determinant of R_s insures that R_s will remain positive definite, independent of the trajectory. Hence, we may derive the filter for optimal estimation of the state variations, $\delta \underline{s}$, according to Kalman[2] employing equations (2.35) through (2.37).

With the system defined as

$$\delta \dot{\underline{s}} = F \delta \underline{s} + G_f \underline{u}_f \tag{3.80}$$

where

$$F = \begin{bmatrix} A \\ 0 \end{bmatrix} + B$$

with measurements

$$\underline{y} = H\,\delta\,\underline{s} + [\,G_f \mid G_a\,]\,\underline{u}_m \tag{3.81}$$

where $H = Z + A$,

the optimal estimate of $\underline{s}$ is obtained from integration of

$$\delta\dot{\tilde{\underline{s}}} = F\,\delta\tilde{\underline{s}} + K\,(\underline{y} - (Z + A)\,\delta\tilde{\underline{s}}) \tag{3.82}$$

where

$$K = (\,PH^T + G_f Q_f G_f^{\;T}\,)\,R_s^{\;-1} \tag{3.83}$$

and where the estimation error covariance matrix, $P(t)$, is obtained from

$$\dot{P} = F\,P + P\,F^T + G_f Q_f G_f^{\;T} - K\,R\,K^T \tag{3.84}$$

The additional term, $G_a Q_a G_a^{\;T}$, in the matrix R_s prevents the direct reduction of the matrices F and $G_f Q_f G_f^{\;T}$ in equation (3.84) as was observed in section 2.3.2 (equation (2.71)) in the case of perfect measurements. We note also that the negative term in (3.84) is inversely proportional to the magnitude of the covariance, q_a, of the measurement noise. Hence a large uncertainty in the measurements due to this noise would decrease the effectiveness of the filter. However, as the covariance, q_a, approaches zero, the estimation errors will approach those obtained from the dual filtering system.

Since no justification can be found for the white noise, $\underline{u}_a$, it is difficult to assign a value to the variance of this noise. In order to evaluate the effects of $\underline{u}_s$, however, an arbitrary value will be chosen by assuming that, over a one second time interval, the accumulated velocity outputs of the accelerometers have an RMS uncertainty of 0.01 feet/sec., and that the correlation time, τ_a, associated with this uncertainty is 0.01 second. We then evaluate q_a as

$$q_a = 2\tau_a \left[\text{RMS} \left(\frac{\Delta V}{\Delta t} \right) \right]^2 = (2)(0.01)(0.01)^2$$

$$q_a = 2 \times 10^{-6} \ \text{feet}^2/\text{sec.}^3$$

The effect of an error of .01 feet/sec.^{2} on the accelerations, f_V and f_γ, along the nominal trajectory is shown in Figure 3.13. We note the major effect of this error upon f_V and f_γ to occur during the initial stage of re-entry and again as the vehicle ascends through a ballistic skip; however, during portions of high acceleration, its effect is overcome by the acceleration dependent noises considered in section 3.4.

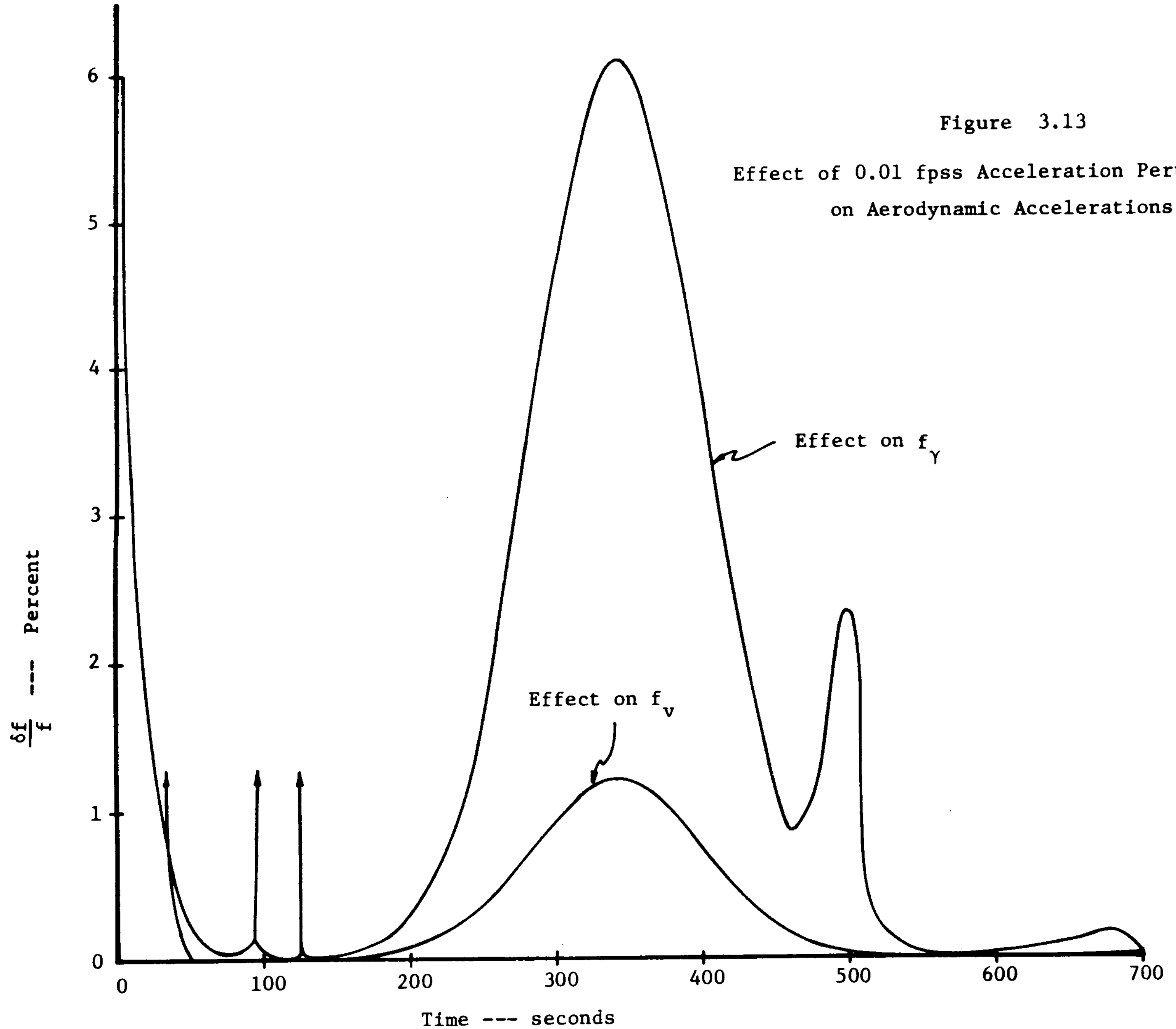


Figure 3.13

Effect of 0.01 fpss Acceleration Perturbations on Aerodynamic Accelerations

CHAPTER IV

COMPUTER SIMULATION OF APOLLO RE-ENTRY NAVIGATION SYSTEM

This chapter presents quantitative results obtained from a computer simulation of inertial navigation systems employed during a typical Apollo re-entry mission.

4.1 Digital Computer Program

The computer program employed to simulate the re-entry navigation systems considered here is shown in Appendix A. The program was written in the MAD language for use on an IBM 7094 computer with a time-sharing facility and provides for numerical integration of the nominal equations of motion for the state variables and of the navigation error covariance matrix P. Integration is performed through a fourth-order Runge-Kutta scheme. The roll control program is obtained from tabular values through a third order Newtonian interpolation scheme.

The program is designed to study three types of navigation systems:

1. A deterministic system for perfect measurements.
2. A dual-filter statistical system employing alternate use of Filters A and B (see section 3.6) with perfect measurements.
3. A single filter statistical system employing only filter A for perfect measurements with additive white noise.

4.2 Vehicle Parameters and Nominal Trajectory

Typical values of parameters for an Apollo command module during re-entry were obtained from B. Crawford[27] as

$$\text{Weight} = 11{,}000 \text{ lb.}$$

$$A_c = \text{Frontal Area} = 129.4 \text{ ft.}^2$$

$$\alpha = \text{Angle of Attack} = 22 \text{ degrees}$$

$$\frac{C_L}{C_D} = \text{Ratio of Lift to Drag} = 0.3$$

The dependence of the lift and drag coefficients on the angle of attack and sideslip angle, as approximated by equations (3.3), cannot be found in the open literature. Without further knowledge of this dependence, we will assume

$$C_{Lo} = C_{Yo} = -C_{L1} = -C_{Y1}$$

and

$$C_{Do} = -C_{D1} = -C_{D2}$$

From the known values of α and C_L/C_D, then, we obtain

$$C_{Lo} = 1.1803$$

and

$$C_{Do} = 1.5107$$

The geodetic parameters assumed in this study are

$$R_e = \text{Earth Radius} = 2.09029 \times 10^7 \text{ feet}$$

$$\Omega = \text{Earth Rotation Rate} = 7.292115 \times 10^{-5} \text{ rad./sec.}$$

$$g_o = \text{Gravitational acceleration at Surface} = 32.2168 \text{ ft./sec.}^2$$

The atmospheric density is approximated as

$$\rho = \rho_o e^{-\beta h}$$

where

$$\rho_o = 2.3769 \times 10^{-3} \text{ slug/ft.}^3$$

and

$$\beta = 4.2553191 \times 10^{-5} \text{ feet}^{-1}$$

The nominal trajectory, assumed to be in an equatorial plane about a spherical rotating earth, is obtained from equation (3.6) with given initial conditions and roll control program. The initial conditions in inertial coordinates were assumed to be

$$V(0) = 36,200 \text{ feet/sec.}$$
$$\gamma(0) = -6.0 \text{ degrees}$$
$$h(0) = 400,000 \text{ feet}$$
$$\theta(0) = 0 \text{ degrees}$$

After conversion to a rotating coordinate system, we obtain

$$V(0) = 34655.5 \text{ feet/sec.}$$
$$\gamma(0) = -6.268 \text{ degrees}$$

The nominal roll control program was obtained from B. Crawford(27) through a computer simulation of the proposed Apollo guidance scheme developed by the MIT Instrumentation Laboratory. The control program, as shown in Figure 4.1, is designed to achieve a range of 2550 N. Miles (on a non-rotating earth) in an equatorial plane with the above initial conditions. The ratio of lift to drag in the plane of motion for this roll program is shown in Figure 4.2.

The nominal trajectory produced by the above initial conditions and roll program is illustrated in Figures 4.2 and 4.3. All values shown here are with respect to a rotating earth. The accelerations

a_V and a_γ (defined by equation (3.16)) for this nominal trajectory are shown in Figure 4.4.

4.3 Initial Covariance Matrix

Uncertainties in estimates of position and velocity at the start of re-entry will be based on navigation errors during the midcourse or trans-earth phase of the mission. A midcourse navigation error analysis has been conducted by G. Levine[28] through a Monte Carlo simulation of a typical Apollo trans-earth trajectory. The navigation system in this analysis utilized celestial sightings to determine the vehicle position and velocity. The results of fifty individual runs obtained from this analysis were employed to compute the statistical properties of initial navigation errors for re-entry. A short computer program was written to convert deviations in inertial position and velocity vectors to deviations in velocity, flight path angle, altitude, and range, and to compute the statistical properties of these deviations. The program and results obtained from it are shown in Appendix B. From the results of this analysis, the following RMS values of initial uncertainties were obtained

$$\mathrm{RMS}(\delta V) = 23.92 \text{ feet/sec.}$$

$$\mathrm{RMS}(\delta\gamma) = 0.183 \text{ degrees}$$

$$\mathrm{RMS}(\delta h) = 27{,}155 \text{ feet}$$

$$\mathrm{RMS}(R_e\delta\theta) = 137{,}090 \text{ feet}$$

with correlation coefficients

$$\rho_{V\gamma} = 0.99519$$

$$\rho_{Vh} = -0.99878$$

$$\rho_{v\theta} = 0.99536$$

$$\rho_{\gamma h} = -0.99367$$

$$\rho_{\gamma\theta} = 0.99998$$

$$\rho_{h\theta} = -0.99411$$

4.4 Results of Computer Simulation

Uncertainties in estimation of position and velocity have been assumed within this study to be random errors with zero mean and with statistical properties described by the covariance matrix

$$P(t) = \mathcal{E}\left[(\delta \underline{s}(t) - \delta \underline{\tilde{s}}(t)) \; (\delta \underline{s}(t) - \delta \underline{\tilde{s}}(t))^T \right]$$

where $\delta \underline{s}(t)$ is the actual variation in the state vector and $\widetilde{\delta \underline{s}}(t)$ is the estimated variation. The differential equations for this covariance matrix have been derived in Chapters II and III for the deterministic and statistical navigation systems. In this section, we show a quantitative comparison of the RMS estimation errors as obtained from these navigation schemes.

4.4.1 Deterministic Navigation with Perfect Measurements

In the deterministic scheme, navigation is performed by direct integration of the specific force acceleration measurements and the computed gravitational accelerations. The differential equation for the estimation error covariance matrix with an assumed perfect IMU system is shown by equation (2.24) to be

$$\dot{\bar{P}} = B\bar{P} + \bar{P}B^T$$

We note the absence of the term $\bar{G} Q_g \bar{G}^T$ here since we have assumed no random disturbances in the gravitational accelerations.

The matrix B is obtained from the linear model of the vehicle dynamics as

$$B = [\, I \; O \,] \; \bar{B} \begin{bmatrix} I \\ \underline{0}^T \end{bmatrix}$$

where $\bar{B}$ is defined by (3.32).

RMS uncertainties in V, γ, h, and range obtained from this simulation are shown in Figures 4.5 through 4.8.

In order to examine the effects of the high correlations in the initial estimation errors, another simulation was made with an initially uncorrelated error matrix, P(0). The results of this simulation are also shown in Figures 4.5 through 4.8. A very pronounced reduction in the errors in estimation of velocity and altitude is observed for the initially correlated errors.

4.4.2 Statistical Navigation with Dual Filtering System

We now show the results of navigating with statistical estimation for assumed random disturbances due to atmospheric variations and vehicle oscillations. The filtering equations for this navigation system were derived in section 3.6. It was found that, due to the insensitivity of measurement variations to noise in the sideslip angle when the sine of the roll angle is zero, that two independent filters are necessary. Propagation of the covariance matrix for filter A during the time when sin ϕ is not equal to zero is obtained by the differential equation (3.33).

When sin ϕ approaches zero, filter B is brought into use. Initial reduction of the covariance matrix is computed from equations

(3.68) and (3.69) and the reduced covariance matrix is computed according to equation (3.67).

The switching logic employed in the computer simulation of this dual filtering system is as follows:

If filter A is in operation at the beginning of an integration step, a check is made at each time considered during integration of that step on the absolute value of sin ϕ. If this value becomes less than a threshold value, sin ϕ_o, the integration is halted and filter B is put into operation at the beginning of the time step.

If filter B is in operation at the beginning of any succeeding integration step and if the value of sin ϕ at that time is greater than or equal to sin ϕ_o, a switch is made to filter A.

Since the initial value of the roll angle in the nominal control program being used is 180 degrees, a switch is immediately made to filter B. Due to the perfect measurement obtained, and to the high correlations between the initial estimation errors, a dramatic reduction is observed in all the estimation errors as shown in Table 4.1. The significance of initial correlations in the estimation errors is noted by applying the same reduction to initially uncorrelated estimation errors. The results of this reduction, also shown in Table 4.1, show a significant change only in the estimation errors for range and flight path angle.

RMS Errors at t = 0	δV fps	$\delta \gamma$ deg.	δh mi.	$R_e \delta \theta$ mi.
Before Measurement	23.92	.183	4.47	22.5
After Measurement Initially Correlated	2.27	.0023	.475	.135
After Measurement Initially Uncorrelated	23.92	.164	4.47	9.87

Table 4.1

Initial Reduction of Estimation Errors
due to Perfect Measurement

This behavior becomes apparent through examination of the matrix Z relating variations in state to variations in the measurements in equation (3.27). Since the state variations $\delta\gamma$ and $\delta\theta$ have an equal but opposite effect on the measurements, a perfectly known linear combination of the measurements should tend to reduce the errors in estimation of $\delta\gamma$ and $\delta\theta$.

Three simulations of the dual filtering system were performed to determine the effects of the initial correlations and of the threshold value, ϕ_o, employed in the switching logic. Values of ϕ_o of 0.1 and 5.0 degrees were studied for the initially correlated case and 5.0 degrees was employed for the simulation of initially uncorrelated errors. The results of these studies are shown in Figures 4.9 through 4.12.

We note additional discontinuous reductions in the estimation errors at times of 38, 120, and 592 seconds as the vehicle is rolled through angles of 0, 180, and 360 degrees, respectively. The initially uncorrelated case does not reveal dramatic discontinuities until significant correlations are encountered.

The choice of ϕ_o is seen to have an almost insignificant effect on the estimation of all state variables except that of altitude for which the effect is seen to be quite large.

4.4.3 Statistical Navigation with Single Filtering System

A single filter navigation system was derived in section 3.7 with the assumption of additive white noise entering through the measurement system. Due to the difficulty in assigning a value to the covariance of this additive noise, two simulations were performed for values of the covariance, q_a, of 2.0×10^{-6} and 5.0×10^{-5} feet^2/sec.^{3}, corresponding to RMS errors in acceleration measurements of 0.01 and 0.05 feet/sec.^{2}, respectively, and to correlation times of 0.01 second.

The results of these simulations are also shown in Figures 4.9 through 4.12. The differences in estimation errors obtained for the two values of q_a are almost indistinguishible over most of the flight. Comparing these results with those obtained with the dual filtering system, we note a continuous, yet equally dramatic reduction of the initial estimation errors. Within a time of 50 seconds, the RMS errors in velocity and altitude are almost identical with those obtained with the perfect measurement system. An increase in these errors is then noted due to the increased effect of the noise, u_a, on the measurements as observed in Figure 3.13. The RMS errors in the flight path angle and range also descend rapidly during the initial 10 seconds, but level off to a value somewhat higher than that obtained initially with the dual filtering system. A steady descent is then observed towards the lower dual filter errors. The rapidly ascending errors in estimation of flight path angle near the end of the simulation is observed in all navigation systems due to the large increase in flight path angle shown in Figure 4.2.

4.5 Numerical Difficulties

A discussion of results obtained through numercial integration of a matrix Ricatti equation such as (3.33) would generally be incomplete without a section devoted to numerical difficulties.

An examination of the negative term in (3.33) shows this term to be inversely proportional to the covariance matrix, R, of the measurement noise, as was observed in section 3.6.3. As R approaches conditions of singularity, the derivative of the matrix P grows rapidly in the negative direction, thus reducing the covariance of estimation errors. Although the matrix P should theoretically remain positive semi-definite, errors encountered through truncation and roundoff within the digital integration scheme force it to become negative definite and henceforth totally unstable. This condition is enhanced when high correlations are present in the covariance matrix.

In order to overcome this difficulty, it was found necessary to reduce the size of the time step employed by the Runge Kutta integration scheme until a stable integration of the covariance matrix was obtained. An empirical time step as a function of time was thus derived for each filter simulation to provide for minimum total computer usage and for stability of the integration. The time steps used for each simulation are shown in Figure 4.13. We note that extremely small time steps were necessary for integration of the single filter with $q_a = 2.0\text{x}10^{-6}$ due to the comparatively low value of the R matrix, Due to the number of integration steps required for stability of the P matrix, the total computer usage for each simulation was extremely high as shown in the table below.

Simulation	Number of Integration Steps	Computer Usage (Minutes)
Deterministic	350	1
Dual Filter--Uncorrelated P(0), $\phi_o = 5^o$	28,000	21
Dual Filter--Correlated P(0), $\phi_o = 5^o$	34,000	26
Dual Filter--Correlated P(0), $\phi_o = .1^o$	52,000	39
Single Filter--$q_a = 2. \times 10^{-6}$	360,000	68
Single Filter--$q_a = 5. \times 10^{-5}$	103,000	43

It is felt that increased stability and decreased computation time could be obtained with the use of double precision and a predictor-corrector type numerical integration scheme such as the Adams-Moulton method within the computer program. The insertion of such into the present program would have required a major revision - not only of the program, but of the coding language - and was hence deemed unfeasible at the time. It is recommended that future studies of such systems be conducted with the above numerical difficulties in mind and that the numerical integration programs be designed accordingly.

4.6 Summary of Results

The results obtained from simulations of the three navigation schemes described above show that, in general, the statistical navigation schemes are effective in providing better accuracy in navigation than the conventional deterministic scheme.

The acute decreases in estimation errors obtained through the perfect measurement with the dual filtering system follow closely the theoretical results suggested in section 3.6.3. These reductions are also found to be highly dependent on the correlations in the estimation errors immediately before the switch to filter B is made.

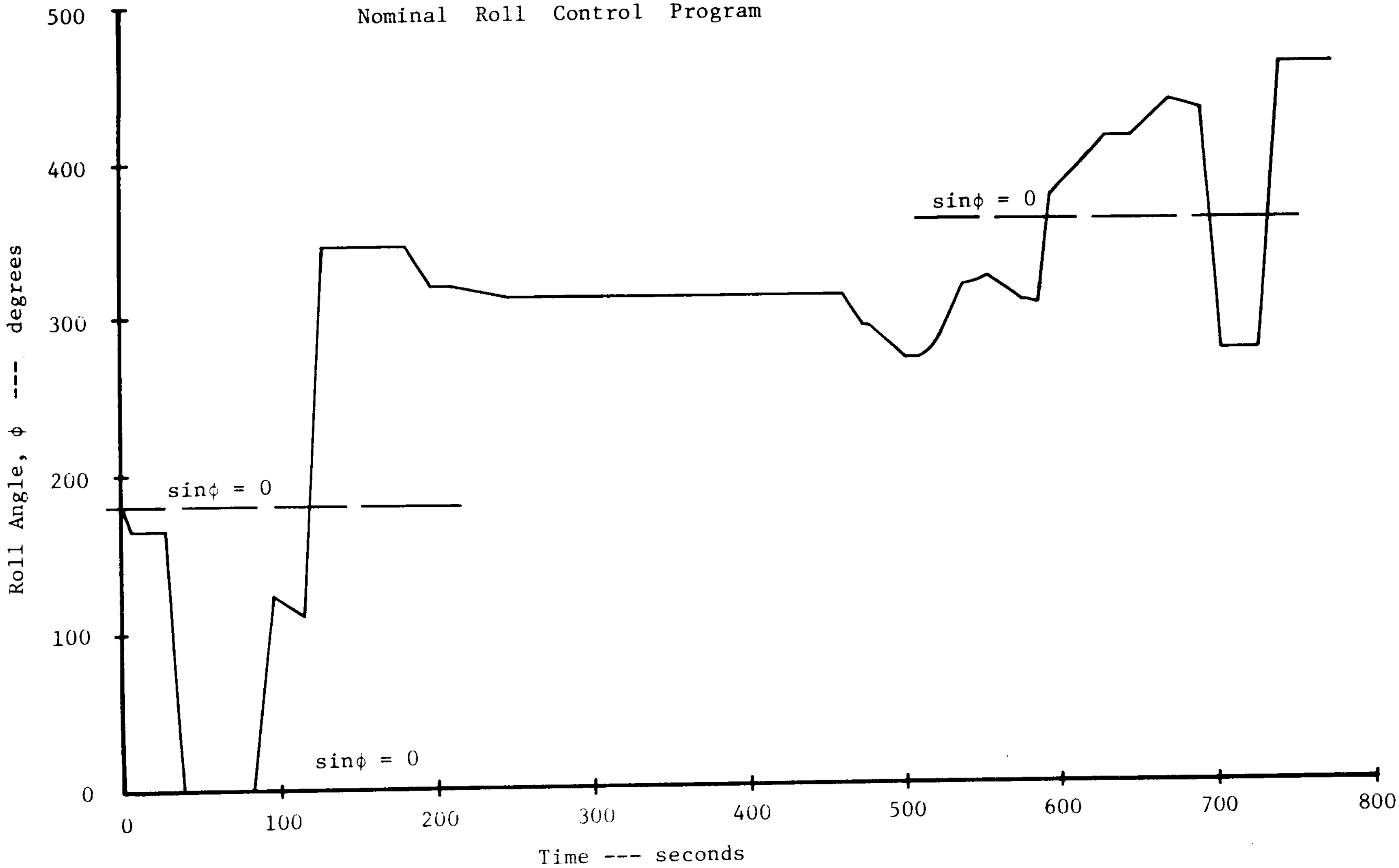
If the initial estimation errors are highly correlated, the results obtained from the dual filtering system would suggest the necessity of statistical filtering only during the first few seconds after the employment of filter B, i.e., after the roll angle reaches 0° or 180°, to reduce the initial estimation errors. After these errors have been sufficiently reduced, the statistical system would provide little advantage over a deterministic one when the measurements contain no white noise. Any additional navigation errors due to bias random errors or colored noise could only be reduced by adding state variables to be estimated by the statistical navigation system. If the initial estimation errors are uncorrelated, however, insufficient information is provided by the perfect measurement obtained when $\sin \phi = 0$ to reduce errors in estimating the velocity and altitude immediately. Thus, the dual filtering statistical navigation system should be employed throughout the flight.

The acceptance of the dramatic reductions exhibited by the dual filtering system would be based on total acceptance of the validity of the assumptions of a measurement system containing no white noise and of white noise entering the aerodynamic forces primarily through angular motions of the vehicle. Due to limited availability of quantitative data concerning random errors present in aerodynamic forces during re-entry and concerning high frequency random errors in inertial measurements, it is impossible at the present time to provide such total acceptance.

The single filtering system includes the effects of an additive white noise entering through the measurement system. Although the results obtained with this system appear to be independent of the magnitude of this noise source within the range of values considered, the accuracy obtained with this system is found to be generally lower than that resulting from the dual filter, yet considerably higher than that derived from a deterministic scheme.

Figure 4.1

Nominal Roll Control Program



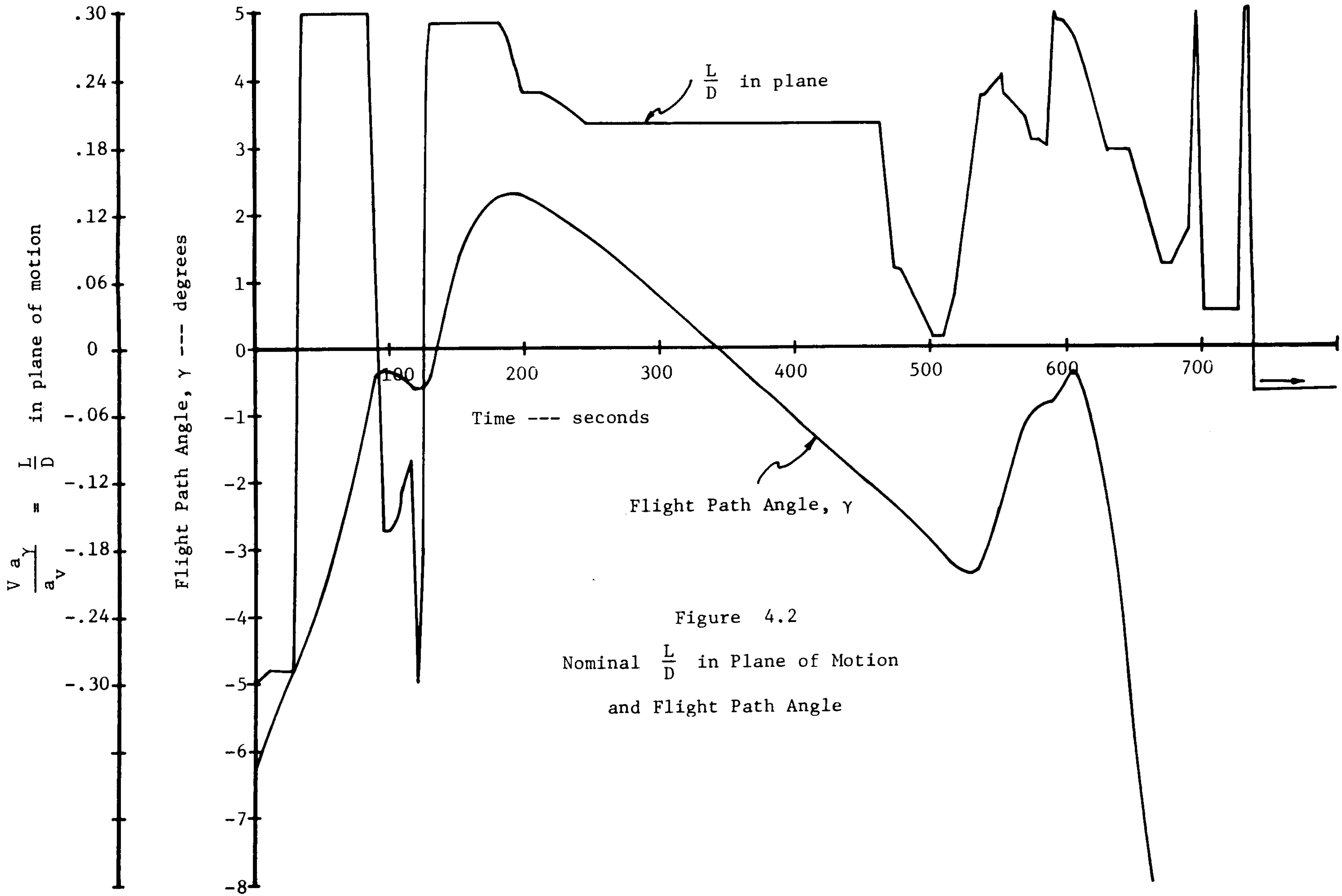


Figure 4.2

Nominal $\frac{L}{D}$ in Plane of Motion

and Flight Path Angle

Figure 4.3

Nominal Velocity, Altitude, and Range

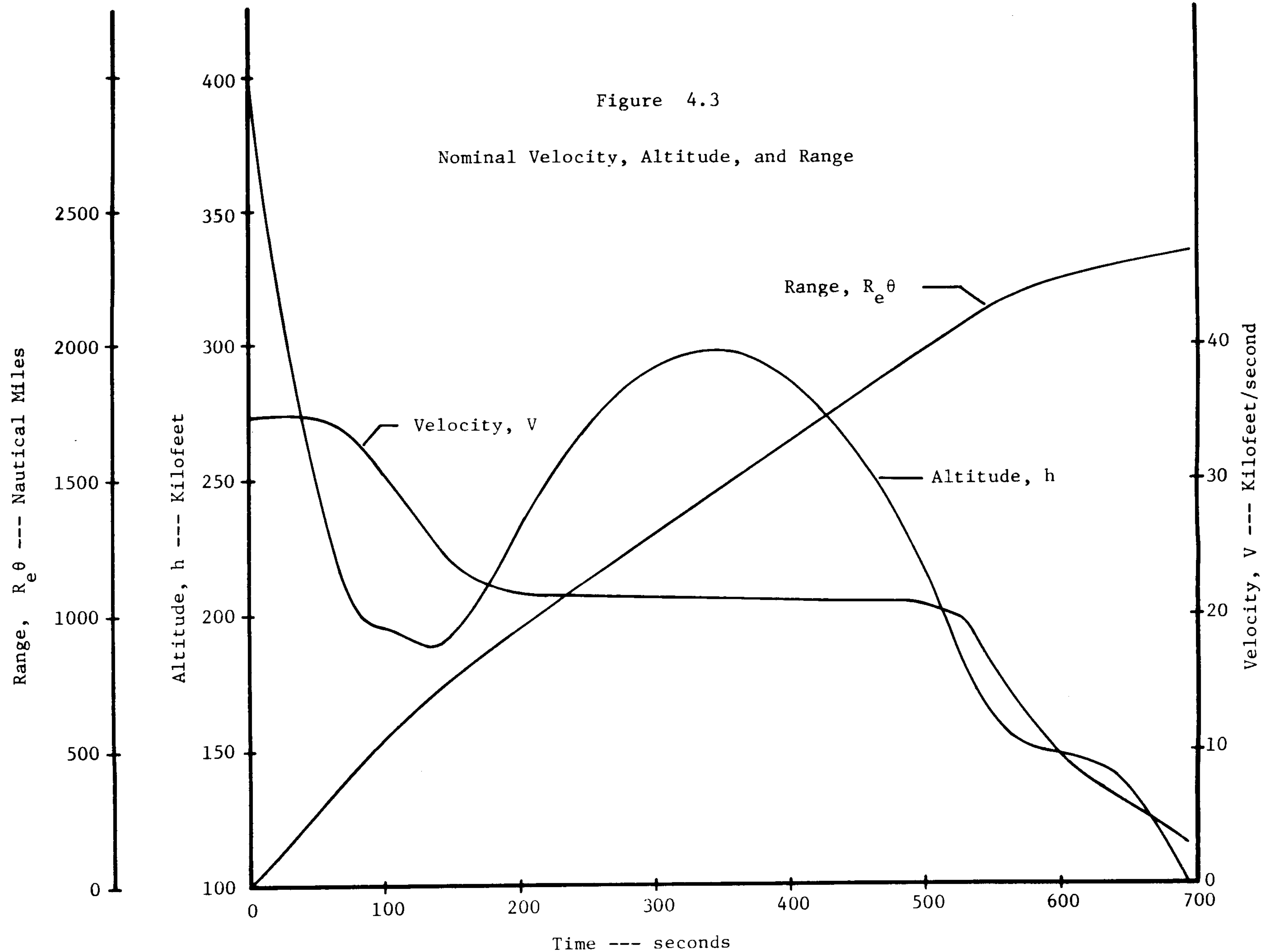


Figure 4.4

Nominal Accelerations

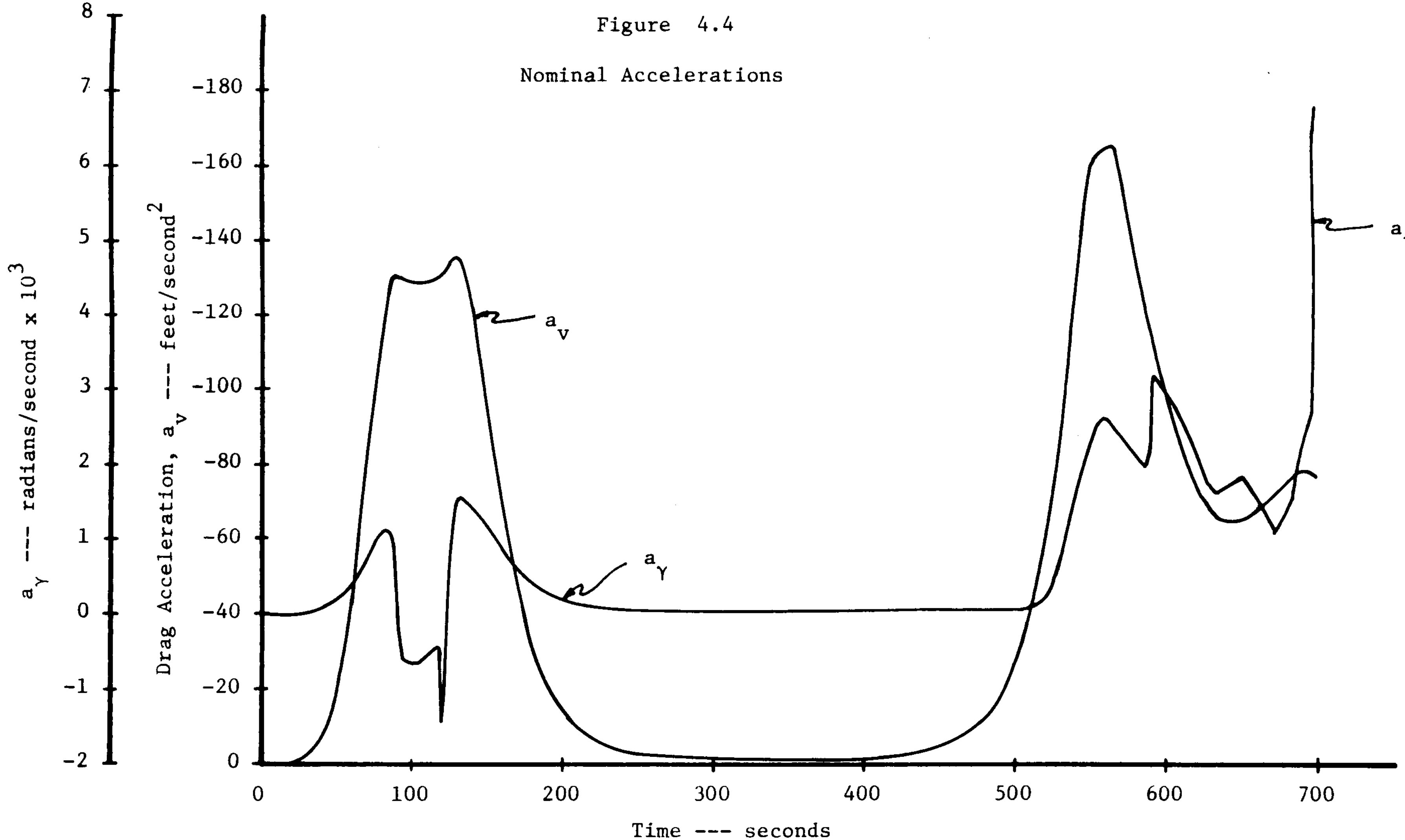
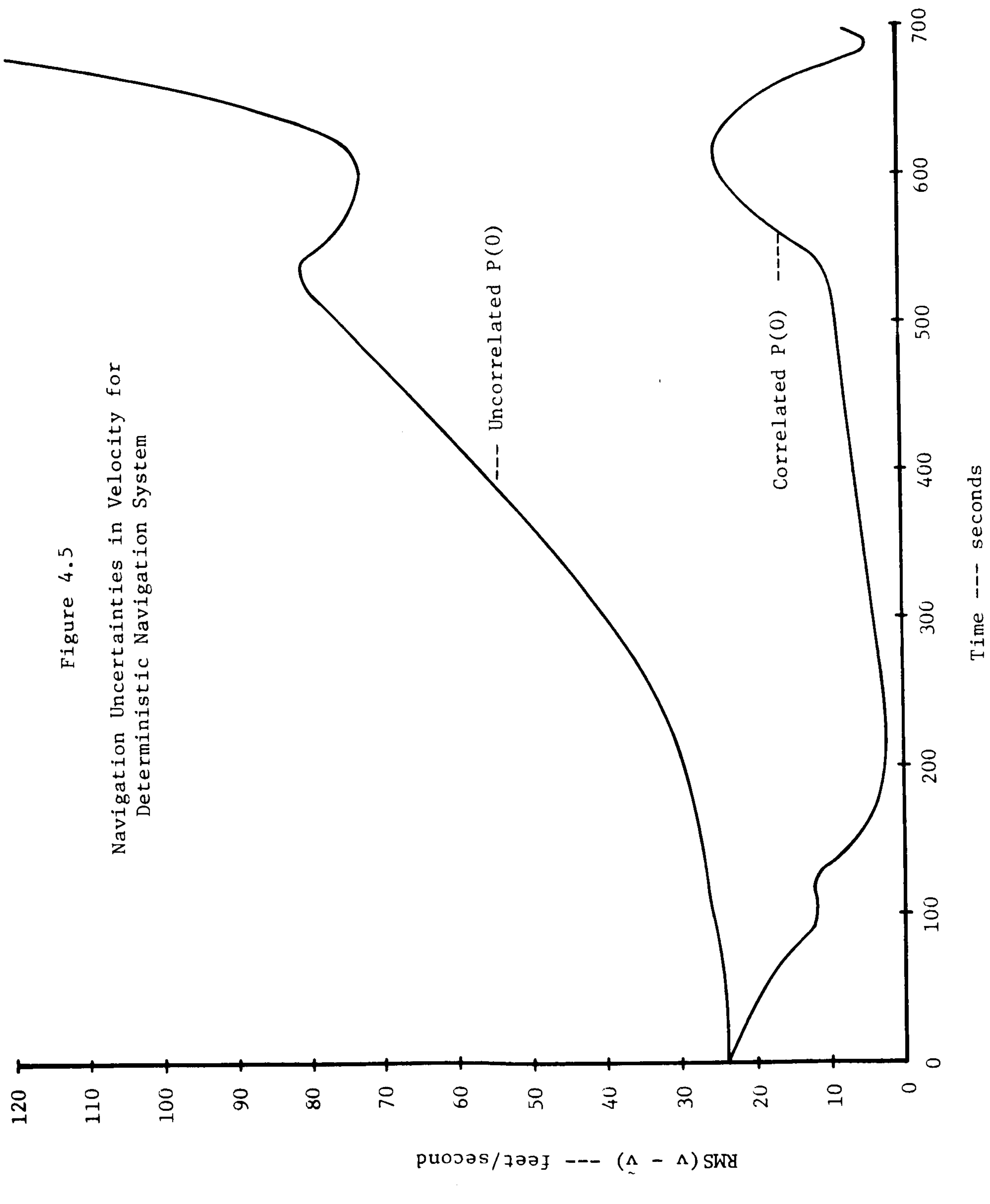


Figure 4.5

Navigation Uncertainties in Velocity for Deterministic Navigation System



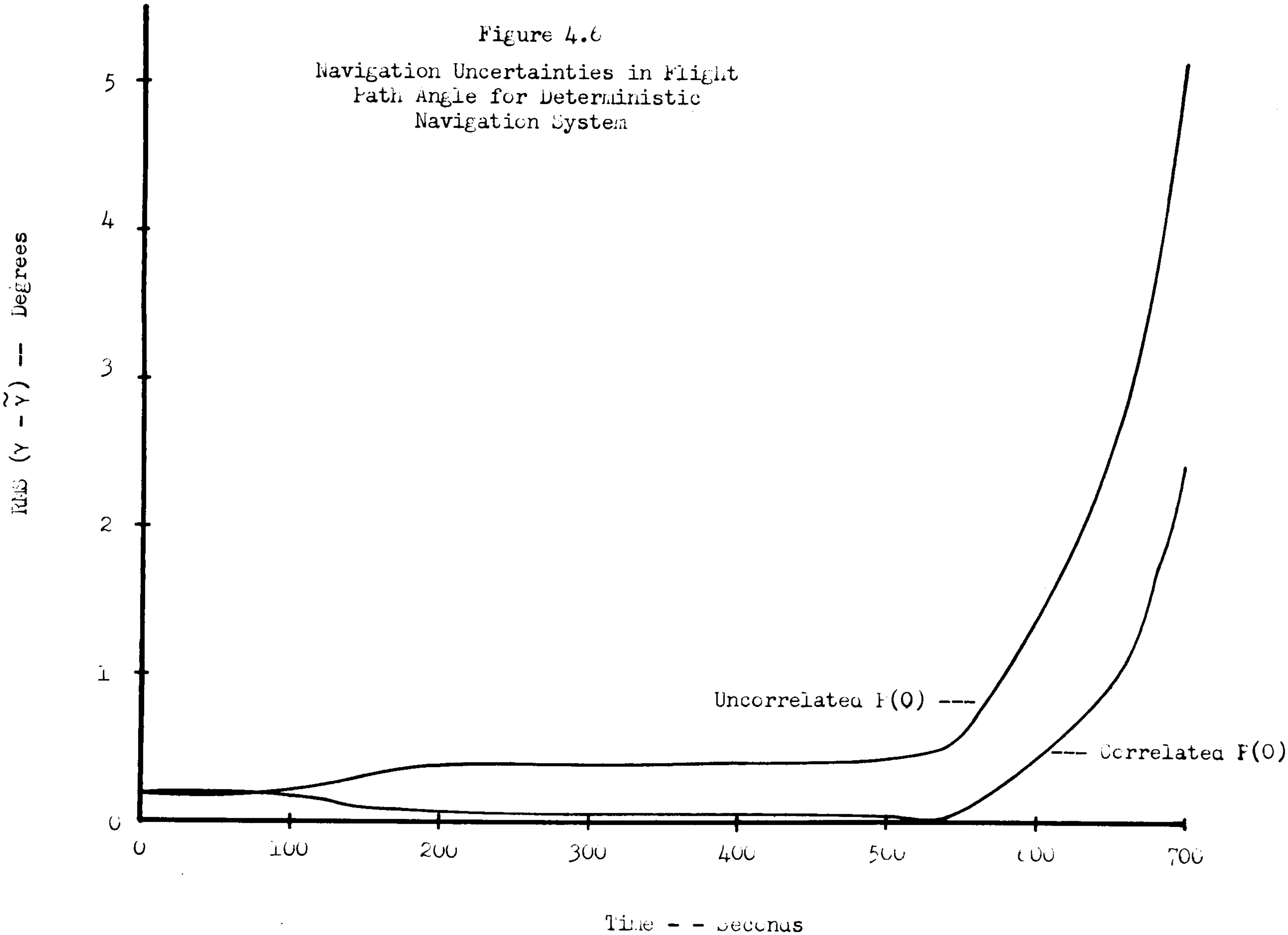
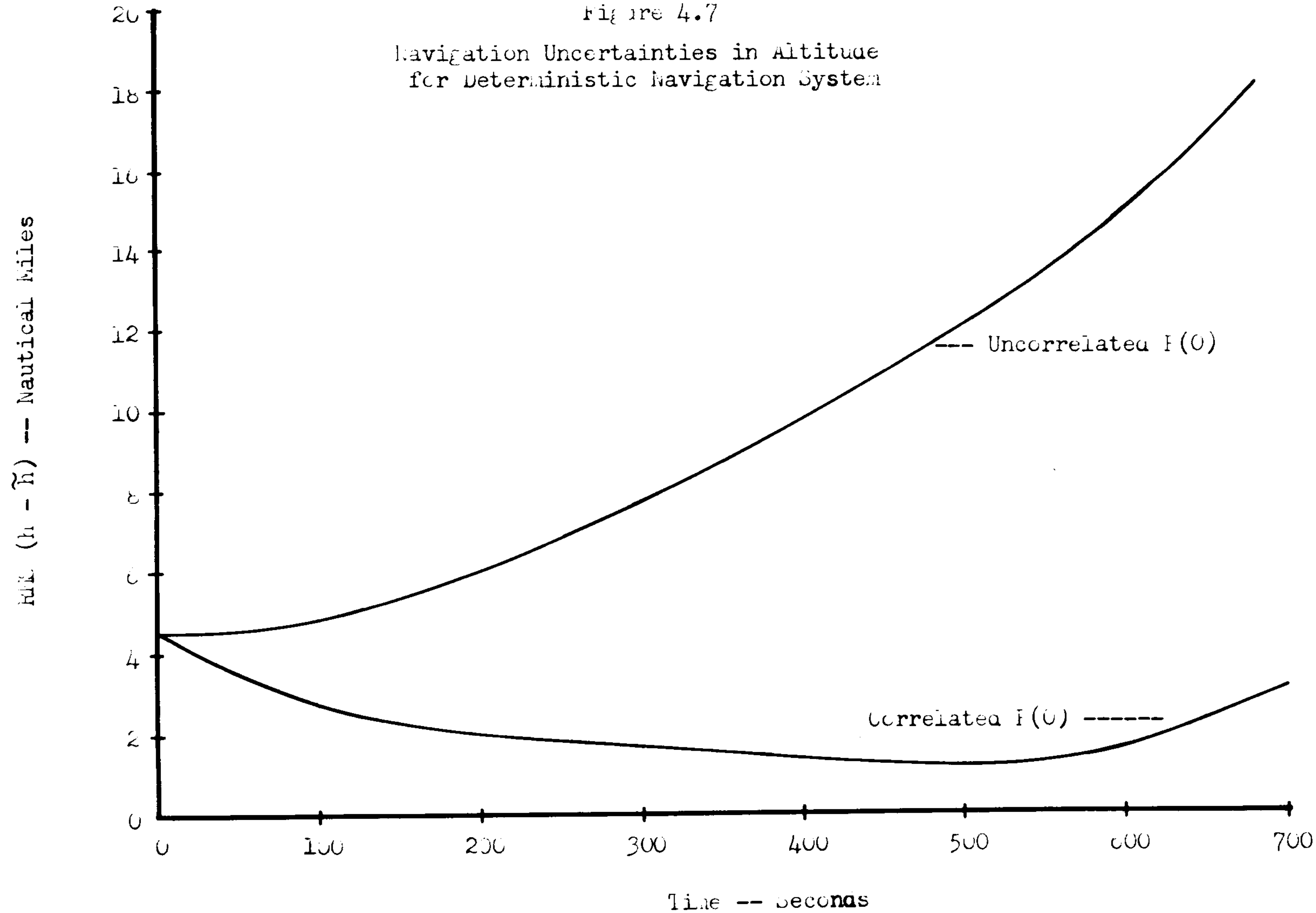


Figure 4.6

Navigation Uncertainties in Flight Path Angle for Deterministic Navigation System

Figure 4.7

Navigation Uncertainties in Altitude
for Deterministic Navigation System



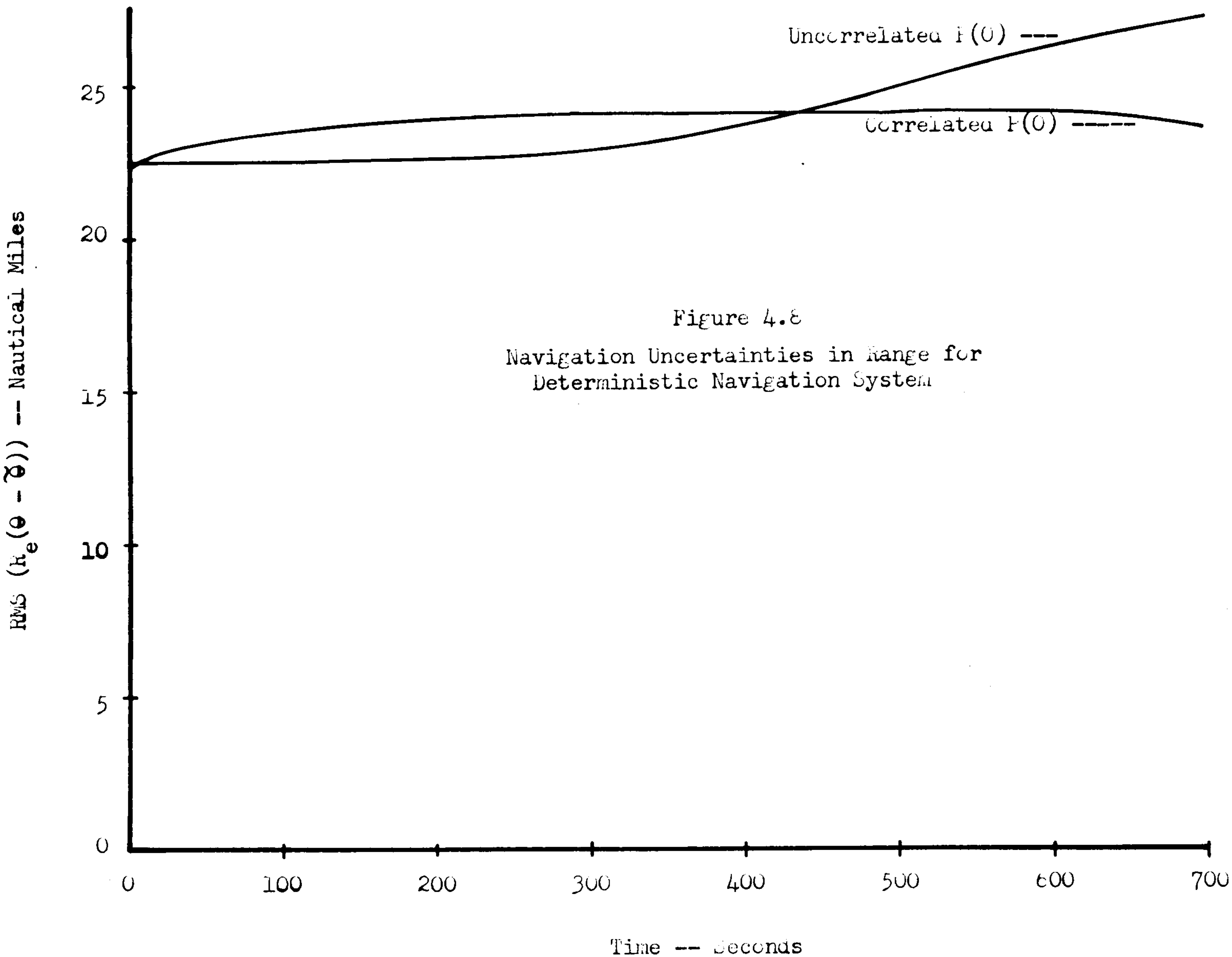


Figure 4.8

Navigation Uncertainties in Range for Deterministic Navigation System

Figure 4.9

Navigation Uncertainties in Velocity for Statistical Navigation Systems

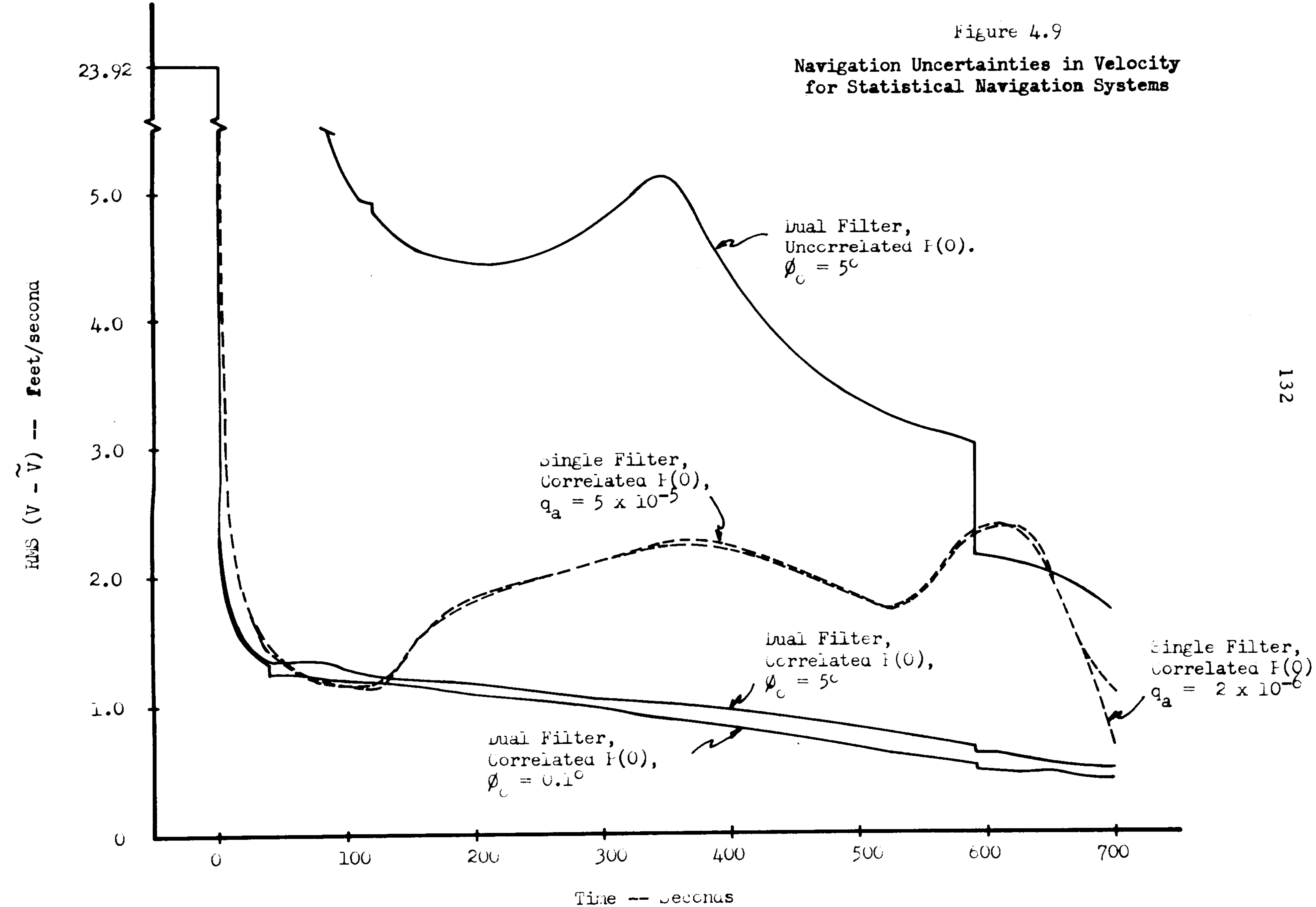


Figure 4.10

Navigation Uncertainties in Flight Path Angle
for Statistical Navigation Systems

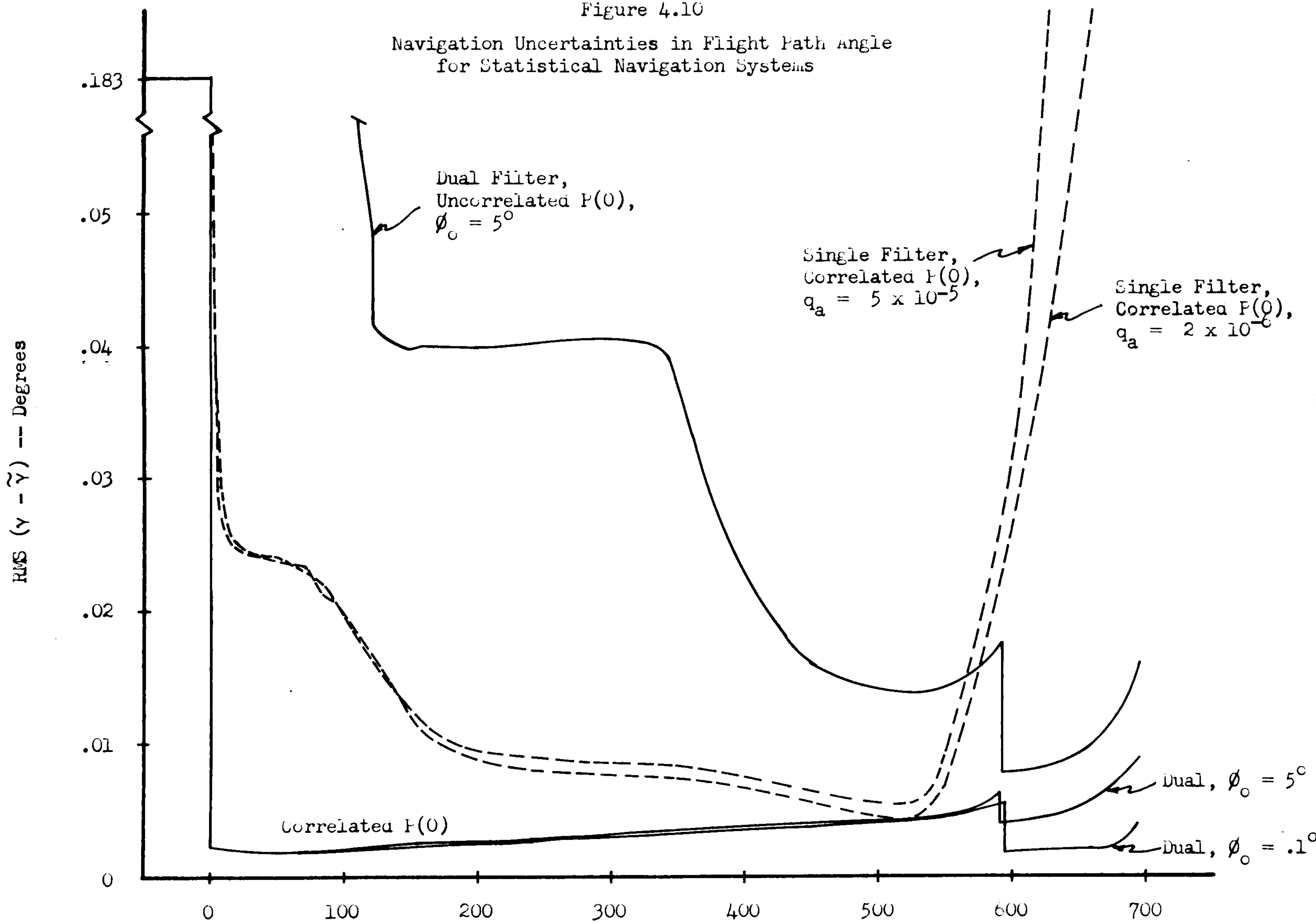
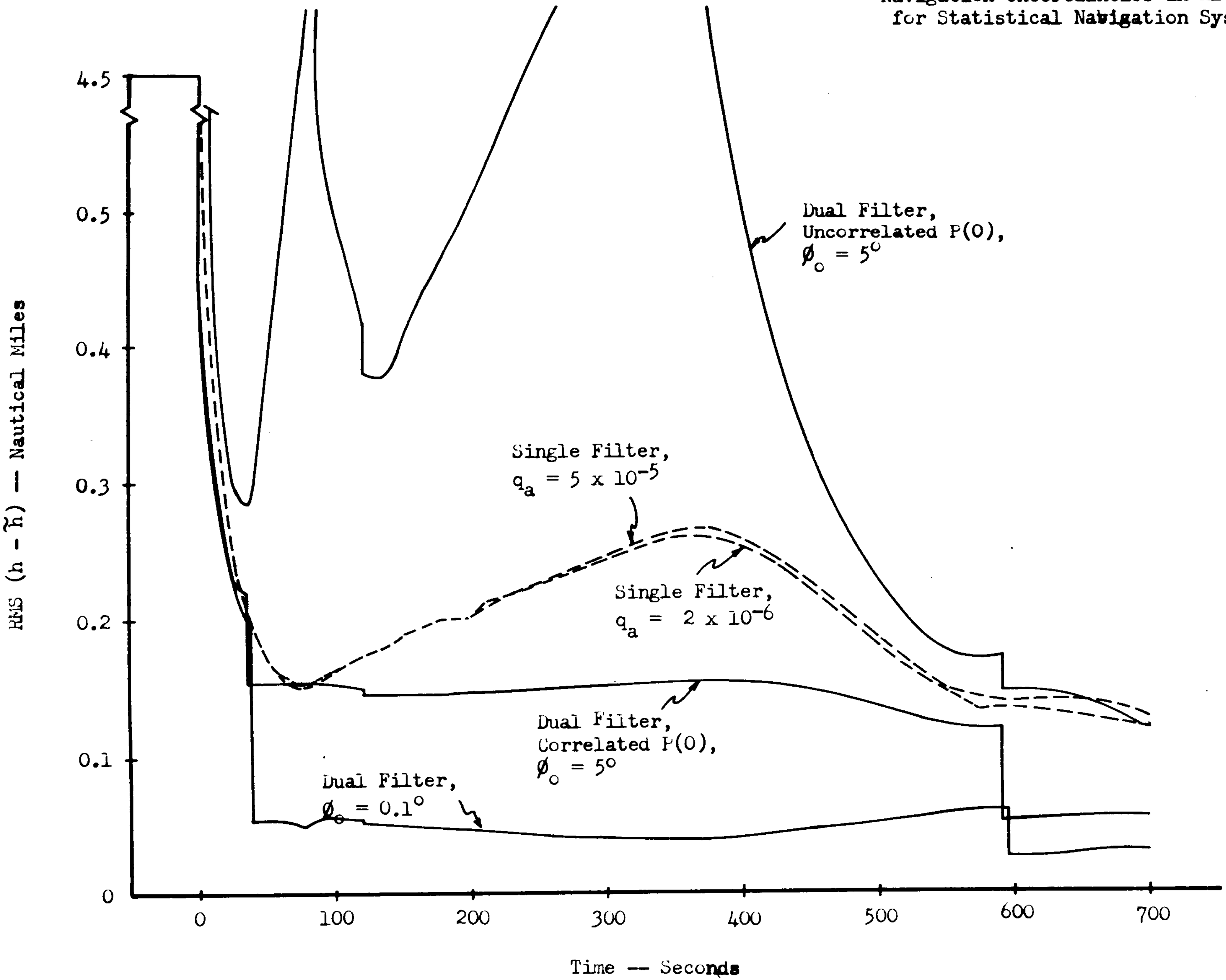


Figure 4.11

Navigation Uncertainties in Altitude for Statistical Navigation Systems



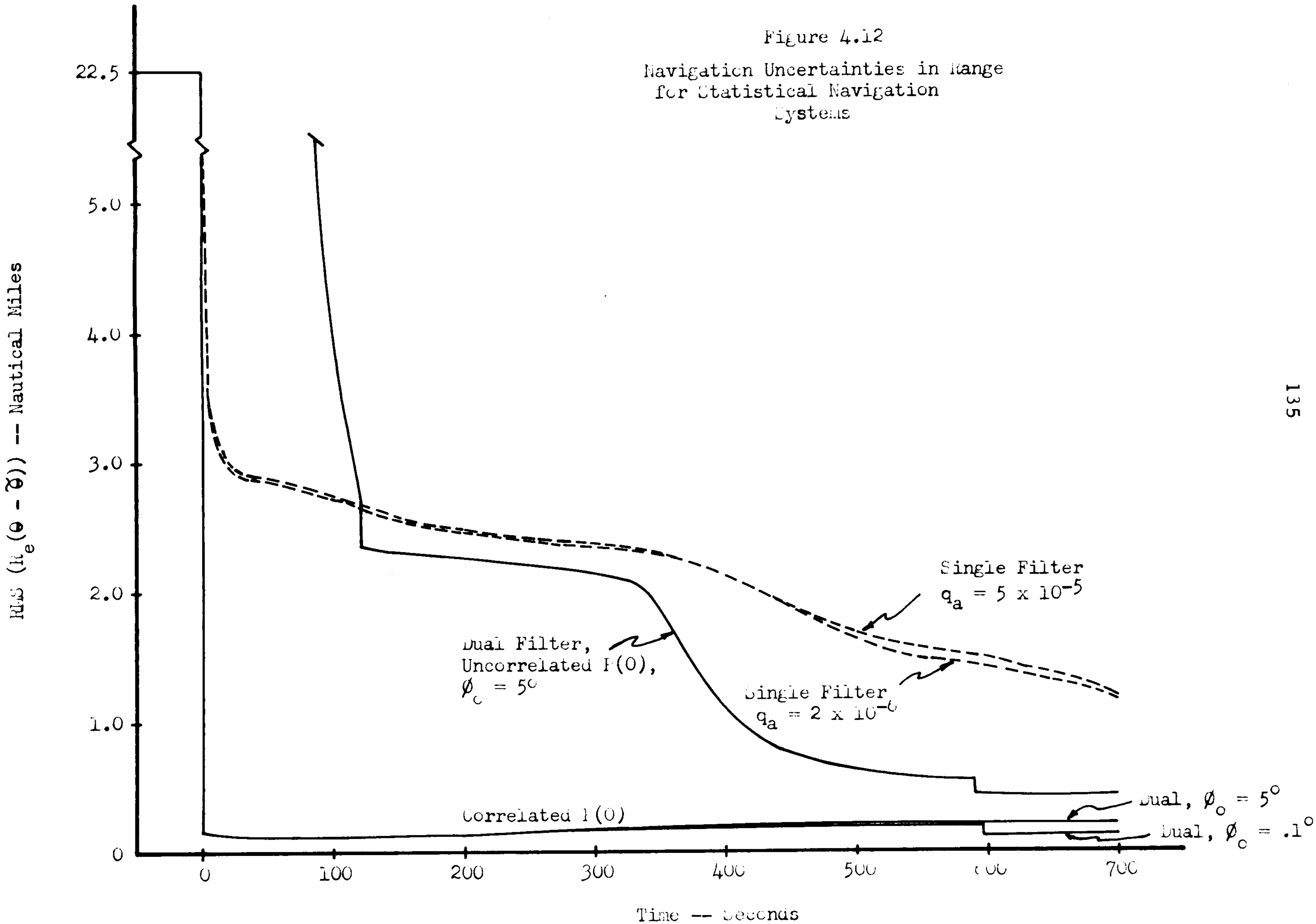


Figure 4.12

Navigation Uncertainties in Range for Statistical Navigation Systems

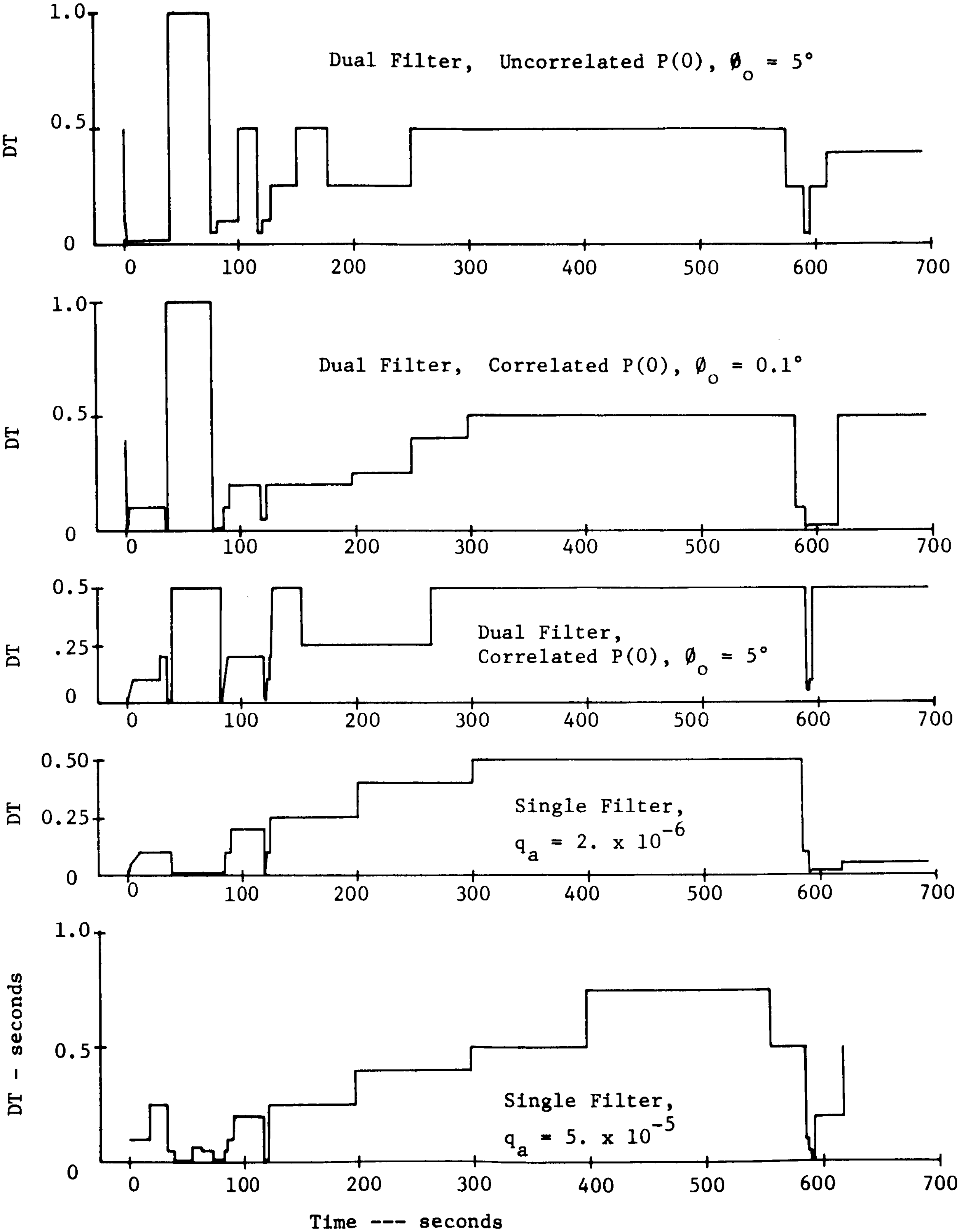


Figure 4.13

Integration Time Steps used in Simulations

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

This research has investigated the statistical inertial navigation of vehicles accelerated primarily by non-gravitational forces during short intervals of time. The source of information for the navigation system is an inertial measurement unit with assumed random constant error coefficients.

Due to the absence of white noise in the accelerometers, the high frequency measurement uncertainties are derived solely from white noise in the specific force accelerations observed by the measurement system. The design and effectiveness of the filter to be incorporated in a statistical navigation scheme was found to be based on the linear dependence of the measurements on these white noise sources. If an insufficient number of white noise elements are present to provide measurements which contain independent white noise elements, then perfect measurements are realized which allow dramatic reductions in estimation errors and which require differentiation before an optimum filtering system can be designed. If the number of independent white noise elements driving the total acceleration of the vehicle is less than or equal to the number of measurements, then the estimation errors in navigation of an observable system are asymptotically reduced to zero when statistical estimation is employed.

Non-gravitational accelerations encountered by vehicles would be derived from propulsive, aerodynamic, or hydrodynamic forces. Due to the independence of thrusting forces on the state of the vehicle, it was found that no advantage could be realized through a statistical navigation scheme during thrusting over the conventional deterministic system. Since aerodynamic forces are dependent on the vehicle's velocity and altitude, however, significant improvement in navigation accuracy can be realized through statistical estimation.

A statistical navigation system was developed for a simplified two-dimensional Apollo re-entry mission. With the assumption that relatively high frequency vehicle oscillations are the primary sources of white noise random errors perturbing the aerodynamic forces, it was found that such a navigation system would require two independent statistical filters for alternate operation depending on the vehicle attitude. Computer simulation for a typical Apollo re-entry trajectory revealed dramatic improvement of navigation accuracy due to the inclusion of statistical estimation in the navigation system. The effectiveness of this improvement was found to be highly dependent on the correlations present in the initial estimation errors.

The dual filtering system, however, was found to be cumbersome and would be rendered impracticable for a real time navigation system due to the necessity for two independent filters, for switching logic, and for the differentiation of measurements. The inclusion of arbitrary additive white noise in the measurement system provided for the derivation of a simplified navigation system requiring a single filter. Results from computer simulation of this single filtering system approached those displayed by the dual system and were found to be independent of the magnitude of the noise source within the range of values considered.

The simplified single filtering system allowed a computation time per integration step of 0.25 second with an IBM 7094 digital computer, as compared with 0.45 second required for the dual system. Due to instability in integration of the estimation error covariance matrix, however, a requirement for smaller integration step sizes forced the total computation time for the single filter simulation to be higher than that required for the dual filtering system.

Unless a more accurate integration scheme is employed which will allow stable numerical integration of the covariance matrix with time steps exceeding the computation time per step, neither the single nor dual system could be recommended as a practicable real-time closed loop on-board navigation system.

An alternative navigation scheme would be an open loop statistical filtering system employing pre-calculated filter gains from assumed nominal roll control programs. No attempt was made within this study to examine the dynamics of the filter gains or to study the effects of employing gains obtained from one nominal trajectory to filter measurements secured along a different trajectory.

A major point made within this thesis is the limited knowledge of the statistics of variations in atmospheric properties within the re-entry altitude range. It is recommended that further study be made to determine more accurately the statistical properties of error sources involved in the prediction and measurement of aerodynamic forces.

Continued effort should be directed toward the design of numerical integration schemes which will provide increased stability of the covariance matrix integration and thus allow the navigation schemes presented herein to be deemed practicable for closed-loop real time navigation. It is also recommended that consideration be extended to the estimation of major error coefficients in the IMU as well as the state of the vehicle.

APPENDIX A

DIGITAL COMPUTER PROGRAM FOR SIMULATION OF APOLLO RE-ENTRY NAVIGATION SYSTEMS

```
       R    MAIN PROGRAM FOR FILTERING OF PERFECT MEASUREMENTS
       R
        PROGRAM COMMON XIN(5),X(5),XX(5),DX(20),PIN(25),P(25),
       1 PP(25),DP(100),C(50),N(30),Y(30),T(10),CT(1000)
        INTEGER I,N,MINT,DELPRT,DUM,SUPPRT,YESORN,NX,NP,DELMES,MEAS
        INTEGER J,L,M,LL,IJ,IL,LJ,LM,NP2,ROTATE
        DIMENSION SIGMA(5), CORRE(10)
       R
       R    ASSIGN INPUT CONSTANTS
       R
START   EXECUTE GETTM.(DATE,RUNTIM)
        V'S WEIGHT = 11000.0
        V'S AREA = 129.4
        V'S CD0 = 1.5107301
        V'S CL0 = 1.1803427
        V'S CD1 = - 1.5107301
        V'S CL1 = - 1.1803427
        V'S EARTHR = 2.090290E7
        V'S ERATE = 0.7292115E-4
        V'S FTTOMI = 6080.2
        V'S RHO = 0.0023769
        V'S BETA = 4.2553191E-5
        V'S G0 = 32.216832
        V'S PI = 3.1415926
        V'S PHI = 0.0
        V'S VEL = 36200.0
        V'S V0 = 25944.042
        V'S VELTH = 0.0
        V'S GAMMA = -6.0
        V'S ALPHA = 22.0
        V'S ALT = 400000.0
        V'S RANGE = 0.0
        V'S K0 = 0.0
        V'S K1 = 0.1
        V'S K2 = 0.2
        V'S MEAS = 2
        V'S SIGMA(1)=23.92,0.183,4.4661426,22.546942,2.0E-11
        V'S CORRE(1)=.99519,-.99878,.99536,0.,-.99367,.99998,0.0
        V'S CORRE(8)=-.99411,0.0,0.0
        V'S SIGAC = 0.01
        V'S SIGALP = 0.0125
        V'S SIGZET = 0.0125
        V'S TIME = 0.0
        V'S T(6) = -1.0
        V'S DELTAT = 2.0
        V'S HMEAS = 410000.0
        V'S ALTFIN = 100000.0
        V'S MINPHI = 5.0
        V'S N(2) = 1
        V'S N(10) = 2
```

```
 V'S NX = 4
 V'S NP = 5
 V'S MINT = 5000
 VECTOR VALUES ROTATE = 1
 V'S DELPRT = 10
 V'S SUPPRT = 1
R
 PRINT FORMAT TITLE, DATE, RUNTIM
 PRINT FORMAT NEWDAT
 READ AND PRINT DATA
 W'R MEAS .E. 2  .AND.  HMEAS .L. 4.0E5, P'T BEGFLT, HMEAS
 V'S BEGFLT = $/H.    FILTERING ONLY BELOW.,F9.1,6HFEET.*$
R
R    COMPUTE CONSTANTS
R
 C(1) = PI/180.0
 C(2) = 180.0/PI
 AL = ALPHA * C(1)
 DT2 = 2.0
 C(9) = AREA * G0 / (2.0 * WEIGHT)
 CD = CD0 + CD1 * AL.P.2
 CL = (CL0 + CL1 * AL.P.2) * AL
 C(3) = C(9) * CD
 C(4) = C(9) * CL
 C(5) = G0
 C(6) = EARTHR
 C(7) = RHO
 C(8) = - BETA * C(6)
 C(10) = 2.0 * CD1 * AL / CD
 C(11) = (CL0 + 3.0*CL1*AL.P.2) / CL
 C(12) = C(10) / C(11)
 C(13) = 1.0 - C(12)
 C(14) = VELTH / C(6)
 C(16) = HMEAS / C(6)
 C(15) = SIN.(MINPHI * C(1))
 C(17) = ALTFIN / C(6)
 C(18) = ((SIGAC/V0).P.2)*.02
 C(20) = 1.0 / FTTOMI
 C(21) = (SIGALP .P. 2) * DT2
 SDT2 = SQRT.(-2.0*C(8))
 C(23) = K0 * SDT2
 C(24) = (4.0*K1 - K2) * SDT2 / 3.0
 C(25) = C(6)*(K2 - K1) * SDT2 / 3.0E5
 C(28) = V0
 C(31) = 2.0 * ERATE
 C(32) = C(6) * ERATE * ERATE
 C(40) = (C(10) * SIGALP) .P.2 * DT2
 C(41) = C(10) * C(11) * SIGALP.P.2 * DT2
 C(42) = ((CD + CL0) * SIGZET / CL).P.2 * DT2
 C(43) = (C(11) * SIGALP) .P. 2 * DT2
```

```
      Y(6) = PHI * C(1)
      Y(7) = COS.(Y(6))
R
R     COMPUTE INITIAL CONDITIONS
R
      WHENEVER ROTATE .E.1
      X(0) = GAMMA * C(1)
      X(3) = ERATE*(C(6) + ALT)/VEL
      X(4) = X(3)/COS.(X(0))
      X(5) = SQRT.(1.0 + X(3)*X(3)*(1.0-2.0/X(4))) * VEL
      X(6) = ASIN.(SIN.(X(0))*VEL/X(5))
      XIN(1) = X(5)/C(28)
      XIN(2) = X(6)
      OTHERWISE
      XIN(1) = VEL / C(28)
      XIN(2) = GAMMA * C(1)
      END OF CONDITIONAL
      XIN(3) = ALT / C(6)
      MILE = C(6) * C(20)
      XIN(4) = RANGE / MILE
      VSIG = SIGMA(1)
      GAMSIG = SIGMA(2)
      HSIG = SIGMA(3)
      RSIG = SIGMA(4)
      SIGRHO = SIGMA(5)
      RHOVG = CORRE(1)
      RHOVH = CORRE(2)
      RHOVR = CORRE(3)
      RHOVP = CORRE(4)
      RHOGH = CORRE(5)
      RHOGR = CORRE(6)
      RHOGP = CORRE(7)
      RHOHR = CORRE(8)
      RHOHP = CORRE(9)
      RHORP = CORRE(10)
      SIGV = .ABS.(VSIG/C(28))
      SIGGAM = .ABS.(GAMSIG*C(1))
      SIGH = .ABS.(HSIG/MILE)
      SIGR = .ABS.(RSIG/MILE)
      T'H PZERO, FOR I = 1,1,I.G.25
PZERO PIN(I) = 0.0
      PIN(1) = SIGV .P. 2
      PIN(2) = SIGV * SIGGAM * RHOVG
      PIN(3) = SIGV * SIGH * RHOVH
      PIN(4) = SIGV * SIGR * RHOVR
      PIN(5) = SIGV*SIGRHO*RHOVP
      PIN(6) = PIN(2)
      PIN(7) = SIGGAM*SIGGAM
      PIN(8) = SIGGAM * SIGH * RHOGH
      PIN(9) = SIGGAM * SIGR * RHOGR
```

```
          PIN(10) = SIGGAM*SIGRHO*RHOGP
          PIN(11) = PIN(3)
          PIN(12) = PIN(8)
          PIN(13) = SIGH * SIGH
          PIN(14) = SIGH * SIGR * RHOHR
          PIN(15) = SIGH * SIGRHO * RHOHP
          PIN(16) = PIN(4)
          PIN(17) = PIN(9)
          PIN(18) = PIN(14)
          PIN(19) = SIGR * SIGR
          PIN(20) = SIGR*SIGRHO*RHORP
          PIN(21) = PIN(5)
          PIN(22) = PIN(10)
          PIN(23) = PIN(15)
          PIN(24) = PIN(20)
          PIN(25) = SIGRHO * SIGRHO
          T(9) = TIME
          T(10) = DELTAT
          N(1) = MINT
          N(11) = DELPRT
          N(13) = NX
          N(14) = NP
          N(9) = MEAS
          N(26) = SUPPRT
          WHENEVER N(2).E.0,TRANSFER TO NOMNAL
          R'T NCARD, N(3)
          T'H READCD, FOR I = 1,1,I.G.N(3)
          J = 8*(I-1) + 4
          R'T CARD, CT(J)...CT(J+7)
READCD    CONTINUE
          V'S NCARD = $I3*$
          V'S CARD = $4(F5.0,E13.8) *$
          N(3) = 8*N(3) + 3
          N(2) = 0
         R
NOMNAL    NP2 = NP * NP
          T(1) = T(9)
          T(2) = T(10)
          T'H INITX, FOR I = 1,1,I.G.NX
INITX     X(I) = XIN(I)
          T'H INITP, FOR I = 1,1,I.G.NP2
INITP     P(I) = PIN(I)
          EXECUTE RUNKUT.(0,ERRET)
          EXECUTE DERVIV.(0)
         R
         R   INTEGRATION
         R
          T'H INGRAT, FOR LL = 1,1,LL.G.MINT
          LL = LL
          EXECUTE CDELT.(LL)
```

```
          EXECUTE RUNKUT.(LL,ERRET)
          W'R X(3) .LE. C(17), T'O FINAL
INGRAT    CONTINUE
          PRINT FORMAT FAILED, MINT
ERRET     TRANSFER TO QUIZ
FINAL     T'H ITERAT, FOR I = 1,1,I.G.5
          W'R .ABS.(C(17)-X(3)) .LE. 5.E-8, T'O EXACT
          DHDT = C(28)*X(1)*Y(8)/C(6)
          T(2) = (C(17) - X(3)) / DHDT
          N(25) = N(11)
ITERAT    EXECUTE RUNKUT.(LL,ERRET)
EXACT     W'R N(26) .E. 3, T'O QUIZ
          EXECUTE RUNKUT.(-1,ERRET)
          V'S FAILED = $///1H I4,68H  INTEGRATION STEPS PERFORMED WITHO
         1UT SATISFYING STOPPING CONDITION.   *$
QUIZ      PRINT FORMAT ASK
          V'S ASK = $///52H CONTROL RETURNED TO MAIN PROGRAM AND RUN CO
         1MPLETED. /45H DO YOU WISH TO CONTINUE THROUGH ANOTHER RUN
         2 /*$
          READ FORMAT ANSWER,YESORN
          V'S ANSWER = $C6 *$
          W'R YESORN .E. $YES    $, T'O START
          EXECUTE EXIT.
          V'S TITLE = $70H1 STUDY OF NAVIGATION SYSTEM WITH FILTERING O
         1F PERFECT MEASUREMENTS    2C7 *$
          V'S NEWDAT = $H.0 ***FOLLOWING IS ALL NEW DATA SUBMITTED FOR
         1THIS RUN ***./*$
          E'M
```

```
R         SINGLE FILTER INTEGRATING ROUTINE
R             SUBROUTINE RUNKUT
R
           EXTERNAL FUNCTION (STEPNO)
           PROGRAM COMMON XIN(5),X(5),XX(5),DX(20),PIN(25),P(25),
          1 PP(25),DP(100),C(50),N(30),Y(30),T(10),CT(1000)
           INTEGER I,J,K,N,LL,LL1,NX,NX2,NX3,NENT,STEPNO,NORTAB
           INTEGER M,NP,NP2,NP22,NP23,MEAS
           DIMENSION DT(4), TAB(8), R(3)
           ENTRY TO RUNKUT.
           W'R STEPNO .G.0
              T'O ENTRY
           O'R STEPNO .L. 0
              T'O PRINT
           OTHERWISE
              T'O BEGIN
           END OF CONDITIONAL
BEGIN      DT(2) = 0.5
           DT(3) = 0.5
           DT(4) = 1.0
           MILE = C(6)*C(20)
           NX = N(13)
           NP = N(14)
           NP2 = NP * NP
           NP22 = 2 * NP2
           NP23 = 3 * NP2
           MEAS = N(9)
           W'R X(3) .G. C(16), N(9) = 0
           NX2 = 2 * NX
           NX3 = 3 * NX
           W'R N(10) .NE. 2, T'O AUX
           T'H TABLE, FOR I = 1,1,I.G.4
TABLE      TAB(I) = CT(I+3)
           NENT = 7
AUX        EXECUTE AUXLRY.
           N(25) = 0
           EXECUTE CDELT.(0)
           W'R N(26) .G. 1, T'O RETURN
           PRINT FORMAT PGSKIP
           PRINT FORMAT HEAD1
           P'T XOUT, T(1),X(1)*C(28),X(2)*C(2),X(3)*C(6),X(4)*MILE,
          1 Y(6)*C(2),Y(5)
           PRINT FORMAT HEAD2
           T'H DIAGON, FOR I = 1,1,I.G.5
           J = 5*(I-1) + I
DIAGON     DP(I) = SQRT.(P(J))
           P'T SIGMA, DP(1)*C(28),DP(2)*C(2),DP(3)*MILE,DP(4)*MILE,DP(5)
           I = 1
           J = 1
           T'H CROS, FOR K = 1,1,K.G.10
```

```
              J = J + 1
              W'R J .LE. 5, T'O GOONN
              I = I + 1
              J = I + 1
GOONN         LL = 5*(I-1) + J
              DP = DP(I) * DP(J)
CROS          DP(K+5) = P(LL) / DP
              P'T CORRE, DP(6)...DP(15)
              F'N
ENTRY         T'H XINIT, FOR I = 1,1,I.G.NX
XINIT         XX(I) = X(I)
              T'H PINIT, FOR I = 1,1,I.G.NP2
PINIT         PP(I) = P(I)
             R
             R   COMPUTE DERIVATIVES
             R
              EXECUTE DERVIV. (1)
              T(3) = T(1)
              T'H LOOP, FOR VALUES OF LL = 2,3,4
              DEL = T(2) * DT(LL)
              T(1) = T(3) + DEL
              LL1 = NX * (LL - 2)
              T'H XSTEP, FOR I = 1,1,I.G.NX
XSTEP         X(I) = XX(I) + DX(LL1+I) * DEL
              EXECUTE AUXLRY.
              LL1 = NP2*(LL-2)
              T'H PSTEP, FOR I = 1,1,I.G.NP2
PSTEP         P(I) = PP(I) + DP(LL1+I) * DEL
             R
              EXECUTE DERVIV. (LL)
LOOP          CONTINUE
              DEL = DEL / 6.0
              T'H NEWX, FOR I = 1,1,I.G.NX
NEWX          X(I) = XX(I)+(DX(I)+2.*(DX(I+NX)+DX(I+NX2))+DX(I+NX3))*DEL
              EXECUTE AUXLRY.
              W'R X(3) .G. C(16)
                 N(9) = 0
              O'E
                 N(9) = MEAS
              E'L
              T'H NEWP, FOR I = 1,1,I.G.NP2
NEWP          P(I) = PP(I) + (DP(I) + 2.*(DP(I+NP2) + DP(I+NP22))
             1                 + DP(I+NP23)) * DEL
              THROUGH TSTPNG, FOR I = 1,6,I.G.25
              W'R P(I) .L. 0.0, T'O PRINT
TSTPNG        CONTINUE
             R
              WHENEVER T(1) .L. T(6), FUNCTION RETURN
              N(25) = N(25) + 1
              W'R N(25) .E. N(11), T'O PRINT
```

```
RETURN       FUNCTION RETURN
PRINT        N(25) = 0
             W'R N(26) .G. 2, F'N
             PRINT FORMAT XOUT, T(1),X(1)*C(28),X(2)*C(2),X(3)*C(6),
            1          X(4)*MILE,Y(6)*C(2),Y(5)
             T'H DIAG, FOR I = 1,1,I.G.5
             J = 6*I - 5
DIAG         DP(I) = SQRT.(P(J),NEGP)
             P'T SIGMA, DP(1)*C(28),DP(2)*C(2),DP(3)*MILE,DP(4)*MILE,DP(5)
             V'S SIGMA = $1HO 5E18.8 *$
             I = 1
             J = 1
             T'H CROSS, FOR K = 1,1,K.G.10
             J = J + 1
             W'R J .LE. 5, T'O GOON
             I = I + 1
             J = I + 1
GOON         LL = 5*(I-1) + J
             DP = DP(I) * DP(J)
CROSS        DP(K+5) = P(LL)/DP
             P'T CORRE, DP(6)...DP(15)
             V'S CORRE = $1H 10F12.8,/*$
             F'N
NEGP         P'T PNEG, I, P(1)...P(25)
             V'S PNEG = $//31H P MATRIX DIAGONAL ELEMENT NO.  I3,
            1 21H HAS BECOME NEGATIVE.  /29H THE ENTIRE P MATRIX FOLLOWS.
            2    // 5(5E16.8,/) *$
             ERROR RETURN
            R
            R
            R    SUBROUTINE AUXLRY
            R
            R
             INTERNAL FUNCTION
             ENTRY TO AUXLRY.
             Y(4) = 1.0 + X(3)
             Y(0) = C(5) / Y(4).P.2
             Y(5) = C(7) * EXP.(C(8)*X(3))
             Y(8) = SIN.(X(2))
             Y(9) = COS.(X(2))
             Y(15) = C(23) + (C(24)+C(25)*X(3)) * Y(5)
CONTRL       W'R T(1).LE.TAB(3) .OR. NENT .GE. N(3), T'O INTERP
             TAB(1) = TAB(3)
             TAB(2) = TAB(4)
             TAB(3) = CT(NENT+1)
                NENT = NENT + 2
             TAB(4) = CT(NENT)
             TRANSFER TO CONTRL
INTERP       Y(6)=TAB(2)+(T(1)-TAB(1))*(TAB(4)-TAB(2))/(TAB(3)-TAB(1))
             Y(7) = COS.(Y(6))
```

```
 R = C(28) * Y(5) * X(1)
 Y(16) = - C(3) * R * X(1)
 Y(10) = C(4) * R
 Y(17) = Y(10) * Y(7)
 Y(1) = - Y(10) * SIN.(Y(6))
 F'N
 E'N
R
R
 V'S XOUT = $1H F9.5,F13.3,F14.7,F13.2,F15.6,F10.2,E12.4 *$
 V'S PGSKIP = $//1H S20,30HINTEGRATION OF STATE VARIABLES*$
 V'S HEAD1= $///1H S2,4HTIMES8,4HVEL.S10,5HGAMMAS8,4HALT.
1S11,5HRANGES6,3HPHIS7,3HRHO / 1H S2,4HSEC.S8,5HFPS. S9,
2 4HDEG.S9,3HFT.S12,5HN.MI.S6,4HDEG. S3,10HSL./CU.FT./*$
 V'S HEAD2 = $1H0S10,26HERROR COVARIANCE MATRIX, P  /*$
R
 END OF FUNCTION
```

```
R    SINGLE FILTERING OF PERFECT MEASUREMENTS
R       SUBROUTINE DERVIV
R
      EXTERNAL FUNCTION (LL)
      PROGRAM COMMON XIN(5),X(5),XX(5),DX(20),PIN(25),P(25),
     1 PP(25),DP(100),C(50),N(30),Y(30),T(10),CT(1000)
      INTEGER I,J,L,M,N,IJ,IL,LM,LJ,JL,LL,LLX,LLP,NP,NP2
      DIMENSION F(25),F(25),H(10),K(10),PH(10),Q(4),R(4),R1(4)
R
      ENTRY TO DERVIV.
      W'R LL .G. 0, T'O ENTRY
      NP = N(14)
      NP2 = NP * NP
      THROUGH FZERO, FOR I = 1,1,I.G.25
FZERO F(I) = 0,0
      F'N
ENTRY X1 = C(28) * X(1)
      FV = Y(16)
      FG = Y(17)
      Y(22) = Y(4) * C(32) / C(28)
      Y(21) = Y(8) * Y(22)
      Y(22) = Y(9) * Y(22) / X(1)
      Y(24) = - Y / C(28)
      Y(23) = Y(24) * Y(8)
      Y(24) = Y(24) * Y(9) / X(1)
      SUM1 = Y(21) + Y(23)
      SUM2 = Y(22) + Y(24)
      LLX = 4*(LL-1)
      DX(LLX+1) = FV + SUM1
      DX(LLX+4) = X1 * Y(9) / (Y(4) * C(6))
      DX(LLX+2) = FG + DX(LLX+4) + C(31) + SUM2
      DX(LLX+3) = X1 * Y(8) / C(6)
      A = (Y(22) - 2.0*Y(24)) / Y(4)
      Q2 = Y(15)*/(15)*(.ABS.(DX(LLX+3)) + C(14))
R
R    COMPUTE F MATRIX
R
      F(11) = DX(LLX+3) / X(1)
      F(12) = DX(LLX+4) * Y(4)
      F(16) = DX(LLX+4) / X(1)
      F(17) = - DX(LLX+3) / Y(4)
      F(18) = - DX(LLX+4) / Y(4)
      F(1) = 2*FV/X(1)
      F(2) = SUM2 * X(1)
      F(3) = (Y(21) - 2.0*Y(23))/Y(4) + C(8)*FV
      F(5) = FV / Y(5)
      F(6) = F(16) + (FG-SUM2)/X(1)
      F(7) = F(17) - SUM1/X(1)
      F(8) = A + F(18) + C(8)*FG
      F(10) = FG / Y(5)
```

```
          F(25) = C(8) * .ABS.(DX(LLX+3))
         R
          Q(1) = FV*FV*C(40)
          Q(2) = FV*FG*C(41)
          Q(3) = Q(2)
          Q(4) = FG*FG*C(43) + C(42)*Y(1)*Y(1)
          LLP = NP2*(LL-1)
         R
         R
          T'H PDOT1, FOR I = 1,1,I.G.NP
          M = NP*(I-1)
          T'H PDOT1, FOR J = 1,1,J.G.NP
          LM = NP*(J-1)
          IJ = M + J + LLP
          DP(IJ) = 0.0
          T'H PDOT1, FOR L = 1,1,L.G.NP
          IL = M + L
          LJ = NP*(L-1) + J
          JL = LM + L
PDOT1     DP(IJ) = DP(IJ) + F(IL)*P(LJ) + P(IL)*F(JL)
          DP(LLP+1) = DP(LLP+1) + Q(1)
          DP(LLP+2) = DP(LLP+2) + Q(2)
          DP(LLP+6) = DP(LLP+6) + Q(3)
          DP(LLP+7) = DP(LLP+7) + Q(4)
          DP(LLP+25) = DP(LLP+25) + Q2
         W'R N( D .EC 0, F'N
          R
          R
           R(1) = Q(1) + C(18)
           R(2) = Q(2)
           R(3) = Q(3)
           R(4) = Q(4) + C(18)/X(1).P.2
           A1 = R(1)*R(4) - R(2)*R(3)
           R1(1) = R(4) / A1
           R1(2) = -R(3) / A1
           R1(3) = R1(2)
           R1(4) = R(1) / A1
           H(1) = 2.0 * FV / X(1)
           H(2) = - FG * X(1)
           H(3) = C(8) * FV
           H(4) = - H(2)
           H(5) = FV / Y(5)
           H(6) = 2 * FG / X(1)
           H(7) = FV / X(1)
           H(8) = C(8) * FG
           H(9) = - H(7)
           H(10) = FG / Y(5)
          R
           THROUGH PHT, FOR I = 1,1,I.G.5
           LM = 2*I
```

```
          PH(LM-1) = 0.0
          PH(LM) = 0.0
          THROUGH PHT, FOR J = 1,1,J.G.5
          IJ = 5*(I-1) + J
          PH(LM-1) = PH(LM-1) + P(IJ)*H(J)
PHT       PH(LM) = PH(LM) + P(IJ)*H(J+5)
          THROUGH ADDGS, FOR I = 1,1,I.G.4
ADDGS     PH(I) = PH(I) + Q(I)
          THROUGH FORMK, FOR I = 1,1,I.G.5
          IJ = 2*I
          K(IJ-1) = PH(IJ-1)*R1(1) + PH(IJ)*R1(3)
FORMK     K(IJ) = PH(IJ-1)*R1(2) + PH(IJ)*R1(4)
          THROUGH KRKT, FOR I = 1,1,I.G.5
          THROUGH KRKT, FOR J = 1,1,J.G.5
          IJ = 5*(I-1) + J
          IL = 2*I
          LM = 2*J
KRKT      E(IJ) = K(IL-1)*(R(1)*K(LM-1) + R(2)*K(LM)) + K(IL)*(R(3)*
         1 K(LM-1) + R(4)*K(LM))
         R
         R   FINAL DERIVATIVE OF P MATRIX
         R
          THROUGH PDOT2, FOR I = 1,1,I.G.NP2
PDOT2     DP(LLP+I) = DP(LLP+I) - E(I)
          F'N
          E'N
```

```
R       DUAL FILTER INTEGRATING ROUTINE
R               SUBROUTINE RUNKUT
R
         EXTERNAL FUNCTION (STEPNO)
         PROGRAM COMMON XIN(5),X(5),XX(5),DX(20),PIN(25),P(25),
        1 PP(25),DP(100),C(50),N(30),Y(30),T(10),CT(1000)
         INTEGER I,J,K,N,LL,LL1,NX,NX2,NX3,NENT,STEPNO,NORTAB
         INTEGER M,NP,NP2,NP22,NP23,COUNT,L,IJ,IK,KL,LJ
         INTEGER MEAS
         DIMENSION DT(4), TAB(8), R(3)
         ENTRY TO RUNKUT.
         W'R STEPNO .G.0
            T'O ENTRY
         O'R STEPNO .L. 0
            EXECUTE PRINTS.(FRRET)
            F'N
         OTHERWISE
            T'O BEGIN
         END OF CONDITIONAL
BEGIN    DT(2) = 0.5
         DT(3) = 0.5
         DT(4) = 1.0
         MILE = C(6)*C(20)
         C(18) = 0.0
         NX = N(13)
         NP = N(14)
         NP2 = NP * NP
         NP22 = 2 * NP2
         NP23 = 3 * NP2
         MEAS = N(9)
         WHENEVER X(3) .G. C(16), N(9) = 0
         NX2 = 2 * NX
         NX3 = 3 * NX
         W'R N(10) .NE. 2, T'O AUX1
         T'H TABLE, FOR I = 1,1,I.G.4
TABLE    TAB(I) = CT(I+3)
         NENT = 7
AUX1     EXECUTE AUXLRY.(PRT)
PRT      N(25) = 0
         EXECUTE CDELT.(0)
         W'R N(26) .G. 1, T'O AUX2
         PRINT FORMAT PGSKIP
         PRINT FORMAT HEAD1
         P'T XOUT, T(1),X(1)*C(28),X(2)*C(2),X(3)*C(6),X(4)*MILE,
        1 Y(6)*C(2),PHIDOT*C(2),Y(5)
         PRINT FORMAT HEAD2
         T'H DIAGON, FOR I = 1,1,I.G.5
         J = 5*(I-1) + I
DIAGON   DP(I) = SQRT.(P(J))
         P'T SIGMA, DP(1)*C(28),DP(2)*C(2),DP(3)*MILE,DP(4)*MILE,DP(5)
```

```
              I = 1
              J = 1
              T'H CROS, FOR K = 1,1,K.G.10
              J = J + 1
              W'R J .LE. 5, T'O GOONN
              I = I + 1
              J = I + 1
GOONN         LL = 5*(I-1) + J
              DP = DP(I) * DP(J)
CROS          DP(K+5) = P(LL) / DP
              P'T CORRE, DP(6)...DP(15)
AUX2          EXECUTE AUXLRY.(SQUASH)
              F'N
SQUASH        EXECUTE REDUCE.(ERRET)
              F'N
ENTRY         W'R N(14).E.4.AND..ABS.(Y(2)).GE.C(15),EXECUTE XPAND.(ERRET)
              COUNT = 0
INITAL        T'H XINIT, FOR I = 1,1,I.G.NX
XINIT         XX(I) = X(I)
              T'H PINIT, FOR I = 1,1,I.G.NP2
PINIT         PP(I) = P(I)
              COUNT = COUNT + 1
              W'R COUNT .G. 3, T'O WRONG
             R
             R    COMPUTE DERIVATIVES
             R
              EXECUTE DERVIV. (1)
              T(3) = T(1)
              T'H LOOP, FOR VALUES OF LL = 2,3,4
              DEL = T(2) * DT(LL)
              T(1) = T(3) + DEL
              LL1 = NX * (LL - 2)
              T'H XSTEP, FOR I = 1,1,I.G.NX
XSTEP         X(I) = XX(I) + DX(LL1+I) * DEL
              EXECUTE AUXLRY.(GOBACK)
              LL1 = NP2*(LL-2)
              T'H PSTEP, FOR I = 1,1,I.G.NP2
PSTEP         P(I) = PP(I) + DP(LL1+I) * DEL
             R
              EXECUTE DERVIV. (LL)
LOOP          CONTINUE
              DEL = DEL / 6.0
              T'H NEWX, FOR I = 1,1,I.G.NX
NEWX          X(I) = XX(I)+(DX(I)+2.*(DX(I+NX)+DX(I+NX2))+DX(I+NX3))*DEL
              EXECUTE AUXLRY.(GOBACK)
              WHENEVER X(3).G.C(16)
                 N(9) = 0
              OTHERWISE
                 N(9) = MEAS
              END OF CONDITIONAL
```

```
                T'H NEWP, FOR I = 1,1,I.G.NP2
NEWP            P(I) = PP(I) + (DP(I) + 2.*(DP(I+NP2) + DP(I+NP22))
               1               + DP(I+NP23)) * DEL
                J = NP + 1
                T'H TSTPNG, FOR I = 1,J,I.G.NP2
                W'R P(I) .L. 0.0, T'O PRINT
TSTPNG          CONTINUE
               R
                W'R T(1) .L. T(6), F'N
                N(25) = N(25) + 1
                W'R N(25) .NE. N(11), F'N
                N(25) = 0
PRINT           EXECUTE PRINTS.(ERRET)
                F'N
GOBACK          EXECUTE REDUCE.(ERRET)
                T'O INITAL
WRONG           PRINT COMMENT $COUNT GREATER THAN 3 IN RUNKUT.$
ERRET           ERROR RETURN
               R
               R
               R    SUBROUTINE PRINTS.
               R
               R
                INTERNAL FUNCTION
                ENTRY TO PRINTS.
                W'R N(26) .G. 2, F'N
                PRINT FORMAT XOUT, T(1),X(1)*C(28),X(2)*C(2),X(3)*C(6),
               1          X(4)*MILE,Y(6)*C(2),PHIDOT*C(2),Y(5)
                T'H DIAG, FOR I = 1,1,I.G.NP
                J = NP*(I-1) + I
DIAG            DP(I) = SQRT.(P(J),NEGP)
                T'O DIAGNL(NP)
DIAGNL(4)       P'T SIGMA, DP(1)*C(28),DP(2)*C(2),DP(3)*MILE,DP(4)*MILE
                V'S SIGMA = $1H0 5E18.8 *$
                KL = 6
                T'O CRSPRD
DIAGNL(5)       P'T SIGMA, DP(1)*C(28),DP(2)*C(2),DP(3)*MILE,DP(4)*MILE,DP(5)
                KL = 10
CRSPRD          I = 1
                J = 1
                T'H CROSS, FOR K = 1,1,K.G.KL
                J = J + 1
                W'R J .LE. NP, T'O GOON
                I = I + 1
                J = I + 1
GOON            LL = NP*(I-1) + J
                DP = DP(I) * DP(J)
CROSS           DP(K+NP) = P(LL)/DP
                P'T CORRE, DP(NP+1)...DP(NP+KL)
                V'S CORRE = $1H 10F12.8 /*$
```

```
          W'R NP .E. 4, PRINT COMMENT $ $
          E'N
NEGP      P'T PNEG, I, P(1)...P(NP2)
          V'S PNEG = $//31H P MATRIX DIAGONAL ELEMENT NO.  I3,
         1 21H HAS BECOME NEGATIVE.  /29H THE ENTIRE P MATRIX FOLLOWS.
         2    // 5(5E15.8,/) *$
          ERROR RETURN
          E'N
         R
         R
         R    SUBROUTINE AUXLRY
         R
         R
          INTERNAL FUNCTION
          ENTRY TO AUXLRY.
          Y(4) = 1.0 + X(3)
          Y(0) = C(5) / Y(4).P.2
          Y(5) = C(7) * EXP.(C(8)*X(3))
          Y(8) = SIN.(X(2))
          Y(9) = COS.(X(2))
          Y(15) = C(23) + (C(24)+C(25)*X(3)) * Y(5)
CONTRL    W'R T(1).LE.TAB(3) .OR. NENT .GE. N(3), T'O INTERP
          TAB(1) = TAB(3)
          TAB(2) = TAB(4)
          TAB(3) = CT(NENT+1)
             NENT = NENT + 2
          TAB(4) = CT(NENT)
          T'O CONTRL
INTERP    PHIDOT = (TAB(4) - TAB(2))/(TAB(3) - TAB(1))
          Y(6) = TAB(2) + PHIDOT * (T(1) - TAB(1))
          Y(7) = COS.(Y(6))
          R = C(28) * Y(5) * X(1)
          Y(16) = - C(3) * R * X(1)
          Y(10) = C(4) * R
          Y(17) = Y(10) * Y(7)
          Y(2) = SIN.(Y(6))
          Y(1) = - Y(10) * Y(2)
          W'R N(14).E.5  .AND.  .ABS.(Y(2)).L.C(15), ERROR RETURN
          W'R N(14).E.4, Y(3) = Y(2) * PHIDOT / Y(7)
          E'N
          E'N
         R
         R
         R    SUBROUTINE REDUCE
         R
         R
          INTERNAL FUNCTION
          E'O REDUCE.
          W'R STEPNO .E. 0, T'O REDUC
          T'H SETX, FOR I = 1,1,I.G.NX
```

```
SETX      X(I) = XX(I)
          T'H SETP, FOR I = 1,1,I.G.25
SETP      P(I) = PP(I)
          P'T PTME, T(1), Y(6)*C(2)
          V'S PTME = $H.0 AT TIME = .,F10.5,H., PHI = .,F6.2*$
          PRINT COMMENT $ CONDITIONS BEFORE SWITCH.$
          T(1) = T(3)
          EXECUTE PRINTS.
REDUC     N(14) = 4
          NP = N(14)
          NP2 = NP * NP
          NP22 = 2 * NP2
          NP23 = 3 * NP2
          EXECUTE AUXLRY.
          A1 = C(13) * Y(16)
          A2 = X(1) * Y(17) / Y(16)
          XX(1) = 2.0 * A1 / X(1)
          XX(4) = A2 + C(12) / A2
          XX(2) = - XX(4)
          XX(3) = C(8) * A1
          XX(5) = A1 / Y(5)
          A1 = 0.0
          T'H CPC, FOR I = 1,1,I.G.5
          DX(I) = 0.0
          T'H PC, FOR J = 1,1,J.G.5
          IJ = 5*(I-1) + J
PC        DX(I) = DX(I) + P(IJ) * XX(J)
CPC       A1 = A1 + XX(I) * DX(I)
          T'H DIVBYD, FOR I = 1,1,I.G.5
DIVBYD    DX(I) = DX(I) / A1
          T'H IMWC, FOR I = 1,1,I.G.5
          T'H IMWC, FOR J = 1,1,J.G.5
          IJ = 5*(I-1) + J
          DP(IJ) = 0.0
          W'R I.E.J, DP(IJ) = 1.0
IMWC      DP(IJ) = DP(IJ) - DX(I) * XX(J)
          T'H NEWPP4, FOR I = 1,1,I.G.4
          T'H NEWPP4, FOR J = 1,1,J.G.4
          IJ = 4*(I-1) + J
          PP(IJ) = 0.0
          T'H NEWPP4, FOR K = 1,1,K.G.5
          IK = 5*(I-1) + K
          T'H NEWPP4, FOR L = 1,1,L.G.5
          KL = 5*(K-1) + L
          LJ = 5*(J-1) + L
NEWPP4    PP(IJ) = PP(IJ) + DP(IK) * P(KL) * DP(LJ)
          T'H NEWP4, FOR I = 1,1,I.G.NP2
NEWP4     P(I) = PP(I)
          P'T CHANGE, T(1)
          V'S CHANGE = $H.0AT TIME = .,F6.1,H., THE COVATIANCE MATRIX D
```

```
           1IMENSION IS REDUCED TO 4.,/*$
            EXECUTE PRINTS.(ERRETR)
            EXECUTE CDELT.(-1)
            F'N
ERRETR      ERROR RETURN
            E'N
           R
           R
           R    SUBROUTINE XPAND
           R
           R
            INTERNAL FUNCTION
            E'O XPAND.
            P'T SWITCH, T(.)
            PRINT COMMENT $ CONDITIONS BEFORE SWITCH.$
            EXECUTE PRINTS.
            N(14) = 5
            NP = N(14)
            NP2 = NP * NP
            NP22 = 2 * NP2
            NP23 = 3 * NP2
            XX(1) = - 2.0 * Y(5) / X(1)
            A1 = X(1) * Y(17) / Y(16)
            XX(2) = Y(5) * (A1 + C(12)/A1) / C(13)
            XX(3) = - C(8) * Y(5)
            XX(4) = - XX(2)
            PP(25) = 0.0
            T'H MPM, FOR I = 1,1,I.G.4
            DX(I) = 0.0
            T'H MP, FOR J = 1,1,J.G.4
            IJ = 4*(I-1) + J
            IK = 5*(I-1) + J
            PP(IK) = P(IJ)
MP          DX(I) = DX(I) + P(IJ) * XX(J)
MPM         PP(25) = PP(25) + XX(I) * DX(I)
            T'H NEWP5, FOR I = 1,1,I.G.25
NEWP5       P(I) = PP(I)
            T'H ADDR5, FOR I = 1,1,I.G.4
            J = I + 20
            P(J) = DX(I)
            J = 5 * I
ADDR5       P(J) = DX(I)
            V'S SWITCH = $H.0AT TIME = .,F10.5,H.,  THE COVARIANCE MATRIX
           1 HAS BEEN EXPANDED TO 5X5.,/*$
            EXECUTE PRINTS.(ERRETX)
            EXECUTE CDELT.(-1)
            F'N
ERRETX      ERROR RETURN
            E'N
           R
```

```
R
 V'S XOUT = $1H F9.5,F13.3,F14.7,F13.2,F15.6,2F10.2,E12.4 *$
 V'S PGSKIP = $//1H S15,H.DUAL FILTERING OF PERFECT MEASUREMEN
1TS.  *$
 V'S HEAD1=$///1H S4,4HTIMES8,4HVEL.S10,5HGAMMAS8,4HALT.
1 S11,5HRANGES6,3HPHIS7,6HPHIDOTS4,3HRHO /1H S4,4HSEC.S8,
2 5HFPS. S9,4HDEG.S9,3HFT.S12,5HN.MI.S6,4HDEG. /*$
 V'S HEAD2 = $1H0 S10,26HERROR COVARIANCE MATRIX, P  /
1 S5,H.FIRST LINE)  RMS ESTIMATION ERRORS IN VEL,GAMMA,ALT,RHO
2,RHO./S5,H.SECOND LINE)  CORRELATION COEFFICIENTS FOR).
3 /S8,H.FILTER A)  V-G,V-H,V-R,V-RHO,G-H,G-R,G-RHO,H-R,H-RHO,R
4-RHO. /S8,H.FILTER B)  V-G,V-H,V-R,G-H,G-R,H-R./*$
R
 END OF FUNCTION
```

```
           R    DUAL FILTERING OF PERFECTMEASUREMENTS
           R       SUBROUTINE DERVIV
            EXTERNAL FUNCTION (LL)
            PROGRAM COMMON XIN(5),X(5),XX(5),DX(20),[illegible],P(25),
           1 PP(25),DP(100),C(50),N(30),Y(30),I(10),C[illegible](1000)
           R
            INTEGER I,J,L,M,N,IJ,IL,LM,LJ,JL,LL,LLX,LLP,NP,NP2
            DIMENSION F(25),F(25),H(10)
           R
            ENTRY TO DERVIV.
            W'R LL .G. 0, T'O ENTRY
            NP = N(14)
            Q1 = 1.0 / C(21)
            E'N
ENTRY       NP = N(14)
            NP2 = NP * NP
            X1 = C(28) * X(1)
            FV = Y(16)
            FG = Y(17)
            Y(22) = Y(4) * C(32) / C(28)
            Y(21) = Y(8) * Y(22)
            Y(22) = Y(9) * Y(22) / X(1)
            Y(24) = - Y / C(28)
            Y(23) = Y(24) * Y(8)
            Y(24) = Y(24) * Y(9) / X(1)
            SUM1 = Y(21) + Y(23)
            SUM2 = Y(22) + Y(24)
            LLX = 4*(LL-1)
            DX(LLX+1) = FV + SUM1
            DX(LLX+4) = X1 * Y(9) / (Y(4) * C(6))
            DX(LLX+2) = FG + DX(LLX+4) + C(31) + SUM2
            DX(LLX+3) = X1 * Y(8) / C(6)
            A = (Y(22) - 2.0*Y(24)) / Y(4)
            Q2 = Y(15)*Y(15)*(.ABS.(DX(LLX+3)) + C(14))
            T'O MESURE(NP)
           R
           R    P MATRIX IS 4-DIMENSIONAL
           R
MESURE(4)   T'H F4ZERO, FOR I = 1,1,I.G.16
F4ZERO      F(I) = 0.0
           R
           R    COMPUTE F MATRIX
           R
            F(9) = DX(LLX+3) / X(1)
            F(10) = DX(LLX+4) * Y(4)
            F(13) = DX(LLX+4) / X(1)
            F(14) = - DX(LLX+3) / Y(4)
            F(15) = - DX(LLX+4) / Y(4)
            F(2) = (SUM2 + FG) * X(1)
            F(3) = (Y(21) - 2.0*Y(23)) / Y(4)
```

```
           F(4) = - FG * X(1)
           F(5) = F(13) - (FG + SUM2) / X(1)
           F(6) = F(14) - DX(LLX+1) / X(1)
           F(7) = A + F(15)
           F(8) = FV / X(1)
          R
          R
           W'R N(9) .E. 0, T'O FINDDP
           A1 = FG / FV
           A2 = X(1) * A1
           A3 = (A2 - C(12)/A2) / C(13)
           A33 = (A2 + C(12)/A2) / C(13)
           A4 = DX(LLX+1) / X(1)
          R
          R    FORM H MATRIX
          R
           H = (A2 + 1.0/A2) / (C(11) - C(10))
           H(1) = Y(5)*(A33*(FG+SUM2)+C(8)*DX(LLX+3)-2.*A4)/X(1)
           H(4) = - Y(5)*(A3*Y(3) + A33*F(8) + 2.0*FG)
           H(2) = - H(4) + Y(5)*(2.0*SUM2 + C(8)*F(10) + A33*SUM1/X(1))
           H(3) = Y(5)*(2.*(F(3)-C(8)*FV)/X(1)-A33*(A-C(8)*FG))
          R
          R    FORM H'Q-1 H PRODUCT
          R
           Q2 = 1.0 / Q2
           T'H HQH, FOR I = 1,1,I.G.4
           T'H HQH, FOR J = 1,1,J.G.4
           IJ = 4*(I-1) + J
HQH        E(IJ) = H(I) * H(J) * Q2
           H = H * H * Q1
           E(6) = E(6) + H
           E(8) = E(8) - H
           E(14) = E(14) - H
           E(16) = E(16) + H
           T'O FINDDP
          R
          R    P MATRIX IS 5-DIMENSIONAL
          R
MESURE(5)  T'H F5ZERO, FOR I = 1,1,I.G.25
F5ZERO     F(I) = 0.0
           F(11) = DX(LLX+3) / X(1)
           F(12) = DX(LLX+4) * Y(4)
           F(16) = DX(LLX+4) / X(1)
           F(17) = - DX(LLX+3) / Y(4)
           F(18) = - DX(LLX+4) / Y(4)
           F(2) = (SUM2 + FG) * X(1)
           F(3) = (Y(21) - 2.0*Y(23)) / Y(4)
           F(4) = - FG * X(1)
           F(6) = F(16) - (FG+SUM2) / X(1)
           F(7) = F(17) - DX(LLX+1) / X(1)
```

```
          F(8) = A + F(18)
          F(9) = FV / X(1)
          F(25) = C(8) * .ABS.(DX(LLX+3))
         R
         R
          W'R N(9) .E. 0, T'O FINDDP
         R
          H(1) = 2.0 * FV / X(1)
          H(2) = - FG * X(1)
          H(3) = C(8) * FV
          H(4) = - H(2)
          H(5) = FV / Y(5)
          H(6) = 2 * FG / X(1)
          H(7) = FV / X(1)
          H(8) = C(8) * FG
          H(9) = - H(7)
          H(10) = FG / Y(5)
          R1 = FV * FV * C(40)
          R2 = FV * FG * C(41)
          R3 = FG*FG*C(43) + C(42)*Y(1).P.2
          A1 = R1 * R3 - R2 * R2
          A2 = R1
          R1 = R3 / A1
          R2 = - R2 / A1
          R3 = A2 / A1
          T'H HRH, FOR I = 1,1,I.G.5
          T'H HRH, FOR J = 1,1,J.G.5
          IJ = 5*(I-1) + J
HRH       E(IJ) = H(I)*(R1*H(J)+R2*H(J+5))+H(I+5)*(R2*H(J)+R3*H(J+5))
         R
         R   COMPUTE DERIVATIVE OF P MATRIX
         R
FINDDP    LLP = NP2*(LL-1)
          T'H PDOT1, FOR I = 1,1,I.G.NP
          M = NP*(I-1)
          T'H PDOT1, FOR J = 1,1,J.G.NP
          LM = NP*(J-1)
          IJ = M + J + LLP
          DP(IJ) = 0.0
          T'H PDOT1, FOR L = 1,1,L.G.NP
          IL = M + L
          LJ = NP*(L-1) + J
          JL = LM + L
PDOT1     DP(IJ) = DP(IJ) + F(IL)*P(LJ) + P(IL)*F(JL)
          W'R NP.E.5, DP(LLP+25) = DP(LLP+25) + Q2
          W'R N(9) .E. 0, F'N
         R
         R   FINAL DERIVATIVE OF P MATRIX
         R
          T'H NEWDP, FOR I = 1,1,I.G.NP
```

```
            T'H NEWDP, FOR J = 1,1,J.G.NP
            IJ = NP*(I-1) + J + LLP
            T'H NEWDP, FOR L = 1,1,L.G.NP
            IL = NP*(I-1) + L
            T'H NEWDP, FOR M = 1,1,M.G.NP
            LM = NP*(L-1) + M
            LJ = NP*(M-1) + J
NEWDP       DP(IJ) = DP(IJ) - P(IL) * E(LM) * P(LJ)
           R
           R
            F'N
            E'N
```

```
R    ROUTINE FOR COMPUTING TIME STEP
R
      EXTERNAL FUNCTION (LL)
      PROGRAM COMMON XIN(5),X(5),XX(5),DX(20),PIN(25),P(25),
1     PP(25),DP(100),C(50),N(30),Y(30),T(10),CI(1000)
R
      DIMENSION Z(50), DT(50)
      INTEGER I,LL,N
      E'O CDELT.
      V'S Z(1) = 90.,118.,120.,140.,230.,450.,1000.
      V'S DT(1) = .01,.1,.05,.1,.2,.5,.5
      W'R LL .NE. 0, T'O ENTRY
      I = 1
      PRINT COMMENT $ NEW VALUES OF Z AND DT FOLLOW $
      READ AND PRINT DATA
      T(2) = DT(I)
      P'T CHGDT, I, T(2)
ENTRY W'R T(1) .LE. Z(I), F'N
      I = I + 1
      T(2) = DT(I)
      P'T CHGDT, I, T(2)
      V'S CHGDT = $/H.  I = ..I2,H..   DT = ..F9.6,/*$
      T'O ENTRY
      E'N
```

APPENDIX B

DIGITAL COMPUTER PROGRAM FOR COMPUTATION OF INITIAL COVARIANCE MATRIX

print covar mad
W 1055.1

COVAR   MAD   08/18 1055.1

```
                    DIMENSION V(50),G(50),H(50),R(50),VH(50),VR(50),RHO(6)
                    INTEGER I
                    :R
                    V'S H = 0.,-7.07,5.01,-3.18,2.32,-6.13,-4.33,11.12,
                    :1 .74,-6.37,.33,-3.96,1.03,-1.19,-8.73,6.03,-1.45,-12.66,
                    :2 -10.52,-1.28,.59,.06,-.1,-.7,7.04,1.21,-.51,-.02,2.37
                    V'S H(29) = -4.87,-2.38,-6.94,7.63,-1.19,4.81,-3.28,-1.63,
                    :1 -5.83,-1.13,3.7,-2.42,-.45,-3.27,3.33,-6.66,12.33,4.88,
                    :2 2.58,2.57,7.27,-4.95
                    :R
                    V'S R(1) = 35.53,-26.16,13.12,-14.91,30.6,21.14,-61.09,-2.95,
                    :1 37.66,-5.82,15.28,-4.46,2.24,40.04,-28.73,8.37,65.27,
                    :2 51.72,7.23,-3.1,-3.78,3.25,5.14,-40.2,-8.95,5.61,.33,
                    :3 -10.39,22.6,13.55,34.05,-42.22,10.86,-18.85,14.75,9.67,
                    :4 27.9,3.18,-17.12,15.02,6.3,21.01,-19.39,33.49,-62.47,
                    :5 -23.94,-10.49,-13.79,-40.48,23.45
                    :R
                    V'S VH(1) = -162.14,118.55,-61.55,67.58,-140.27,-96.6,
                    :1 277.22,13.42,-171.2,25.97,-71.35,20.91,-10.69,-223.27,
                    :2 133.07,-37.26,-299.4,-236.45,-33.,13.8,16.98,-14.64,
                    :3 -23.32,182.2,40.87,-25.11,-1.62,48.2,-103.35,-61.69,
                    :4 -155.24,191.56,-49.52,87.52,-67.51,-43.6,-127.42,
                    :5 -14.62,78.31,-68.08,-28.35,-95.19,88.25,-153.28,
                    :6 284.86,109.48,47.75,62.76,183.71,-106.57
                    :R
                    V'S VR(1) = 15.69,-11.27,8.26,-4.23,13.84,10.07,-24.47,
                    :1 -1.99,11.64,1.03,11.03,-2.63,4.33,17.14,-13.88,2.93,
                    :2 26.41,24.11,2.35,-1.26,1.39,-1.02,.89,-14.7,-1.46,-.15,
                    :3 -.12,-5.96,11.91,4.6,15.75,-16.45,.36,-12.8,7.92,3.12,
                    :4 13.05,3.68,-8.9,4.21,-.73,5.56,-6.17,13.76,-27.81,
                    :5 -11.71,-6.95,-5.75,-14.99,11.99
                    :R
                    C = 0.017453292
                    VEL = 35847.0
                    RAD = 4079.0
                    ERAD = 3963.2
                    GAM = -6.0 * C
                    CS = COS.(GAM)
                    SN = SIN.(GAM)
                    C1 = 0.02
                    MV = 0.0
                    MG = 0.0
                    MH = 0.0
                    MR = 0.0
                    MV2 = 0.0
                    MG2 = 0.0
                    MH2 = 0.0
                    MR2 = 0.0
                    MVG = 0.0
                    MVH = 0.0
                    MVR = 0.0
                    MGH = 0.0
                    MGR = 0.0
                    MHR = 0.0
                    T'H LOOP, FOR I = 1,1,I.G.50
                    RVH = VH(I) / VEL
                    RVR = VR(I) / VEL
                    RH = H(I) / RAD
```

```
          RR = R(I) / RAD
          S1 = SQRT.(1.0 + RH*(RH-2.0) + RR * RR)
          S2 = SQRT.(1.0 - 2.0*(SN*RVH + CS*RVR) + RVH*RVH+RVR*RVR)
          V(I) = VEL * (1.0-S2)
          H(I) = RAD * (1.0-S1)
          RDOTV = (1.0 - RH)*(SN - RVH) - (CS - RVR) * RR
          G(I) = (GAM - ASIN.(RDOTV/(S1*S2)))/C
          R(I) = ERAD * RR / (1.0-RH)
          MV = MV + V(I)
          MG = MG + G(I)
          MH = MH + H(I)
LOOP      MR = MR + R(I)
          MV = MV * C1
          MG = MG * C1
          MH = MH * C1
          MR = MR * C1
          T'H LOOP1, FOR I = 1,1,I.G.50
          V(I) = V(I) - MV
          G(I) = G(I) - MG
          H(I) = H(I) - MH
          R(I) = R(I) - MR
          MV2 = MV2 + V(I)*V(I)
          MG2 = MG2 + G(I)*G(I)
          MH2 = MH2 + H(I)*H(I)
          MR2 = MR2 + R(I)*R(I)
          MVG = MVG + V(I)*G(I)
          MVH = MVH + V(I)*H(I)
          MVR = MVR + V(I)*R(I)
          MGH = MGH + G(I)*H(I)
          MGR = MGR + G(I)*R(I)
LOOP1     MHR = MHR + H(I)*R(I)
          P'T HEAD
          V'S HEAD = $///S10,22HSTATISTICAL ERROR DATA//
          :1 36H        V        GAMMA      ALT          RNG /*$
          T'H PRT, FOR I = 1,1,I.G.50
PRT       P'T OUT, I, V(I), G(I), H(I), R(I)
          V'S OUT = $I2,F9.2,F8.2,2F10.2 *$
          P'T MEAN, MV,MG,MH,MR
          V'S MEAN = $//12H MEAN VALUES  2F8.2,2F10.2 *$
          MV = SQRT.(MV2*C1)
          MG = SQRT.(MG2*C1)
          MH = SQRT.(MH2*C1)
          MR = SQRT.(MR2*C1)
          P'T RMS, MV, MG, MH, MR
          V'S RMS = $/11H RMS VALUES  2F8.3, 2F10.3 *$
          P'T CROSS, MVG*C1,MVH*C1,MVR*C1,MGH*C1,MGR*C1,MHR*C1
          V'S CROSS = $/35H CROSS PRODUCTS (VG,VH,VR,GH,GR,HR)
          :1 /6F10.4 *$
          RHO(1) = MVG*C1/MV/MG
          RHO(2) = MVH*C1/MV/MH
          RHO(3) = MVR*C1/MV/MR
          RHO(4) = MGH*C1/MG/MH
          RHO(5) = MGR*C1/MG/MR
          RHO(6) = MHR*C1/MH/MR
          P'T CORRE, RHO(1)...RHO(6)
          V'S CORRE = $/25H CORRELATION COEFFICIENTS/6F10.5 *$
          E'M
R 2.766+1.483
```

Statistical Analysis of Initial Estimation Errors

```
loadgo covar_ma""
W 1102.4
EXECUTION.
```

STATISTICAL ERROR DATA

	V	GAMMA	ALT	RNG
	FT/SEC	DEG.	MILES	MILES
1	30.14	.23	-6.61	32.01
2	-25.85	-.20	5.54	-27.90
3	12.54	.07	-2.59	10.28
4	-13.39	-.12	2.91	-16.95
5	26.10	.19	-5.63	27.23
6	17.93	.13	-3.77	18.06
7	-56.41	-.44	11.27	-61.97
8	-5.44	-.04	1.35	-5.32
9	27.02	.24	-5.93	34.08
10	-3.76	-.06	.94	-8.11
11	16.30	.09	-3.38	12.38
12	-6.86	-.05	1.64	-6.79
13	3.37	-.00	-.58	-.28
14	37.65	.32	-8.41	45.09
15	-30.01	-.21	6.54	-30.41
16	4.73	.04	-.85	5.67
17	54.29	.42	-12.57	60.77
18	45.88	.34	-10.23	47.67
19	3.72	.03	-.67	4.57
20	-4.75	-.04	1.20	-5.47
21	-2.45	-.04	.67	-6.13
22	-1.54	.01	.51	.70
23	1.26	.02	-.09	2.54
24	-36.17	-.30	7.45	-41.58
25	-7.80	-.08	1.81	-11.15
26	.41	.02	.10	2.99
27	-2.01	-.01	.59	-2.14
28	-13.05	-.09	2.97	-12.56
29	20.45	.14	-4.32	19.48
30	8.92	.08	-1.79	10.70
31	29.51	.22	-6.47	30.57
32	-38.94	-.31	8.02	-43.55
33	3.44	.06	-.59	8.09
34	-24.04	-.15	5.38	-20.79
35	12.82	.08	-2.69	11.86
36	5.58	.05	-1.03	6.94
37	24.02	.17	-5.31	24.61
38	3.13	.01	-.52	.63
39	-19.18	-.13	4.28	-19.10
40	9.18	.09	-1.83	12.13
41	.17	.03	.16	3.66
42	13.30	.13	-2.71	17.94
43	-17.52	-.15	3.90	-21.31

44	27.33	.21	-6.18	30.03
45	-60.58	-.45	12.46	-63.34
46	-25.31	-.18	5.42	-25.74
47	-13.99	-.09	3.18	-12.65
48	-14.39	-.11	3.16	-15.86
49	-36.62	-.30	7.68	-41.86
50	20.85	.14	-4.40	20.30

	ALT	GAMMA	ALT	RANGE
MEAN VALUES	2.06	.02	-.61	2.46
RMS VALUES	23.920	.183	5.143	25.964

CROSS PRODUCTS (VG,VH,VR,GH,GR,HR)
4.3676 -122.8619 618.1644 -.9376 4.7636 -132.7372

CORRELATION COEFFICIENTS
.99519 -.99878 .99536 -.99367 .99998 -.99411

EXIT CALLED. PM MAY BE TAKEN.

R 7.083+1.516

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BIOGRAPHY

Gerald E. Steinker was born on March 23, 1939, in Indianapolis, Indiana. He attended Trinity Lutheran Grammer School and Arsenal Technical High School in Indianapolis and, in July 1956, entered General Motors Institute in Flint, Michigan, through appointment by the Allison Division of General Motors Corporation in Indianapolis.

Mr. Steinker received a B.M.E. degree from G.M.I. in August 1961, and entered MIT to pursue graduate study in the Department of Aeronautics and Astronautics. Efforts being directed in the fields of astronautical guidance and the optimization of trajectories, he received an S.M. degree in January 1963. The results of his master's thesis, entitled "Solutions for Optimal Stochastic Trajectories," was published in the AIAA Journal as a technical note in April 1963.

Mr. Steinker has worked as a research assistant at MIT since September 1961. Some of this time was spent in various departments of the MIT Instrumentation Laboratory, including the Space Guidance Analysis section under the direction of Dr. Richard H. Battin. The remainder of the time was spent on thesis research.

Mr. Steinker is a member of Alpha Tau Iota, Sigma Gamma Tau, and Sigma Xi honorary fraternities. He is also a member of Theta Xi Social Fraternity and, while at MIT, was active in the Graduate Student Council as Aeronautics Representative and Social Chairman and in the Graduate House Executive Committee, as Treasurer.

On December 26, 1964, Mr. Steinker was very happily married to Miss Cheryl Ann Kubiak of Indianapolis. They presently reside in Cambridge, Massachusetts with their daughter, Laurie Ann, who arrived in October 1965.